Recall that the fibers of a map $\varphi : X \to Y$ are the sets in $\varphi^{-1}(y) \subseteq X$ which all map to the same element $y \in Y$.

**Example 1:** Consider the map

$$
\varphi : \mathbb{Z} \to \mathbb{Z}/4\mathbb{Z}
$$

$$
z \mapsto \bar{z}.
$$

The fibers of this map are

\[
4\mathbb{Z} = \{4z \mid z \in \mathbb{Z}\} = \{\ldots, -8, -4, 0, 4, 8, \ldots\} \mapsto \{0\}
\]
\[
4\mathbb{Z} + 1 = \{4z + 1 \mid z \in \mathbb{Z}\} = \{\ldots, -7, -3, 1, 5, 9, \ldots\} \mapsto \{1\}
\]
\[
4\mathbb{Z} + 2 = \{4z + 2 \mid z \in \mathbb{Z}\} = \{\ldots, -6, -2, 2, 6, 10, \ldots\} \mapsto \{2\}
\]
\[
4\mathbb{Z} + 3 = \{4z + 3 \mid z \in \mathbb{Z}\} = \{\ldots, -5, -1, 3, 7, 11, \ldots\} \mapsto \{3\}
\]
Recall that the fibers of a map $\varphi : X \to Y$ are the sets in $\varphi^{-1}(y) \subseteq X$ which all map to the same element $y \in Y$.

**Example 2:** Consider the map $D_{12} \to S_3$ defined by $s \mapsto (12)$, $r \mapsto (123)$.

First, this map extends to a homomorphism: Check

$\varphi(s)^2 = \varphi(1)$, $\varphi(r)^6 = \varphi(1)$, $\varphi(r)\varphi(s) = \varphi(s)\varphi(r)^{-1}$.

Claim: the fibers of this map are

- $K = K = \ker(\varphi) = \{1, r^3\} \mapsto 1$
- $Ks = \{ks \mid k \in K\} = \{s, r^3s\} \mapsto (12)$
- $Krs = \{krs \mid k \in K\} = \{rs, r^4s\} \mapsto (13)$
- $Kr^2s = \{kr^2s \mid k \in K\} = \{r^2s, r^5s\} \mapsto (23)$
- $Kr = \{kr \mid k \in K\} = \{r, r^4\} \mapsto (123)$
- $Kr^2 = \{kr^2 \mid k \in K\} = \{r^2, r^5\} \mapsto (132)$

Note that since $r^3 \in Z(D_{12})$, we have $xK = Kx$ for all $x \in D_{12}$. 
Let $K \leq G$. Then for $g \in G$, we call the sets
\[ gK = \{ gk \mid k \in K \} \quad \text{and} \quad Kg = \{ kg \mid k \in K \} \]
the \textbf{left} and \textbf{right coset} of $K$ (corresponding to $g$).

\textbf{Theorem}

Let $\varphi : G \to H$ be a surjective homomorphism of groups. For each $h \in H$, let $X_h$ be the fiber over $h$:
\[ X_h = \varphi^{-1}(h) = \{ g \in G \mid \varphi(g) = h \} \]
(so, in particular, $X_1 = \ker(\varphi)$).

1. Then
\[ x \in X_a \text{ and } y \in X_b \quad \text{implies} \quad xy \in X_{ab}. \]
In particular as subsets of $G$, $\{ X_h \mid h \in H \}$ is a group under the operation $X_a \ast X_b = X_{ab}$.
(We call this group the \textbf{quotient group} $G/\ker(\varphi)$).

2. Fix some fiber $X_h$. For any $x \in X_h$, we have $X_h$ is both a \textbf{left} and a \textbf{right coset} of the kernel corresponding to $x$:
\[ X_h = \{ xk \mid k \in \ker(\varphi) \} \quad \text{and} \quad X_h = \{ kx \mid k \in \ker(\varphi) \}. \]