Warm-up:

The quaternion group, denoted $Q_8$, is the set

$$\{1, -1, i, -i, j, -j, k, -k\}$$

with product $\cdot$ given by

$$1 \cdot a = a \cdot 1 = a \quad \forall a \in Q_8, \quad (-1) \cdot (-1) = 1,$$

$$i^2 = j^2 = k^2 = -1, \quad (-1) \cdot a = a \cdot (-1) = -a \quad \forall a \in Q_8,$$

(Think: three copies of $\mathbb{C}$)

$$i \cdot j = k = -j \cdot i, \quad j \cdot k = i = -k \cdot j, \quad k \cdot i = j = -i \cdot k.$$  

(Think: cross-product with $i = v_1, j = v_2, k = v_3$)

You try:

1. Write the group table (multiplication table) for $Q_8$.
2. Compute the order of each of the 8 elements of $Q_8$. 
Isomorphisms

Consider the subgroup of $S_6$ generated by

\[ r = (1 6 5 4 3 2) \quad \text{and} \quad s = (16)(25)(34) \]

In some sense, this subgroup is the same as $D_{12}$, but in some sense, they’re not the same until I name them appropriately. Since we don’t want to call them the same, we call them isomorphic.

Let $G$ and $H$ be groups. A homomorphism is a function $\varphi : G \to H$ satisfying

\[ \varphi(g_1 g_2) = \varphi(g_1) \varphi(g_2) \quad \text{“structure preserving”} \quad (*) \]

for all $g_1, g_2 \in G$.

Think: a linear map $L : F^n \to F^m$ of vector spaces “preserves vector space structure”, i.e. for all $v, v' \in F^n$, $c \in F$ (a field),

\[ L(cv) = cL(v) \quad \text{and} \quad L(v + v') = L(v) + L(v'). \]

In a group, the corresponding “structure” is the binary operation. Similarly, a homomorphism of fields would “preserve” both addition and multiplication.

An isomorphism is a bijective homomorphism. Two groups $G$ and $H$ are isomorphic, written $G \cong H$, if there exists an isomorphism between them.
A homomorphism is a function $\varphi : G \to H$ satisfying
\[ \varphi(g_1 g_2) = \varphi(g_1) \varphi(g_2) \quad \text{“structure preserving”} \quad (*) \]
for all $g_1, g_2 \in G$. An isomorphism is a bijective homomorphism.

Examples of isomorphisms:
1. $G$ is always isomorphic to itself via the identity map.
   Note that may be other isomorphisms! An isomorphism $\varphi : G \to G$ is called an automorphism.
2. $(\mathbb{R}, +)$ is isomorphic to $(\mathbb{R}_{>0}, \times)$ via the map $\varphi : x \to e^x$.
   Check: $\varphi$ is a bijection and
   \[ \varphi(x + y) = e^{x+y} = e^x \ast e^y = \varphi(x) \varphi(y). \]
3. $S_X$ is isomorphic to $S_{|X|}$.
4. $S_3$ is isomorphic to $D_6$.

A homomorphism is a function $\varphi : G \to H$ satisfying
\[ \varphi(g_1 g_2) = \varphi(g_1) \varphi(g_2) \quad \text{“structure preserving”} \quad (*) \]
for all $g_1, g_2 \in G$. An isomorphism is a bijective homomorphism.

Examples of homomorphisms that aren’t isomorphisms:
1. Let $\varphi : \mathbb{Z} \to \mathbb{Z}/6\mathbb{Z}$ be given by reducing mod 6.
2. Let $\varphi : \mathbb{Z} \to \mathbb{R}$ be the inclusion map.
3. The determinant map $\det : \text{GL}_n(\mathbb{R}) \to \mathbb{R}^\times$ is a homomorphism.

Fact:

“$G \sim H$ whenever $G \cong H$”
is an interesting equivalence relation. On the other hand,
“$G \sim H$ whenever there’s a homomorphism $\varphi : G \to H$”
is a completely uninteresting equivalence relation.
A homomorphism from a group $G$ to a matrix group $\text{GL}_n(F)$ is called a group representation.

(1) Denote the linear transformation in $\mathbb{R}^2$ that rotates everything clockwise by $\phi$ radians by $r_\phi$ and the linear transformation that flips across the y-axis by $s_y$, i.e.

$$r_\phi = \begin{pmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{pmatrix} \quad \text{and} \quad s_y = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Then $D_{2n}$ is isomorphic to the multiplicative group of matrices generated by $r_{2\pi/n}$ and $s_y$.

(2) The symmetric group $S_n$ is isomorphic to the multiplicative group of $n \times n$ matrices satisfying

every row and column has exactly one 1 and $n - 1$ 0’s.

(Since $S_{\{1,\ldots,n\}} \cong S_{\{v_1,\ldots,v_n\}}$)
Properties of homomorphisms

Theorem

Let $\varphi : G \to H$ be a homomorphism of groups.

1. $\varphi(1_G) = 1_H$.
2. For any $x \in G$, $\varphi(x^{-1}) = \varphi(x)^{-1}$.
3. For any $x \in G$, $|\varphi(x)|$ divides $|x|$.
4. The image of $\varphi$,

$$\text{img}(\varphi) = \{h \in H \mid h = \varphi(g) \text{ for some } g \in G\},$$

is a subgroup of $H$.
5. The kernel of $\varphi$,

$$\text{ker}(\varphi) = \{g \in G \mid \varphi(g) = 1_H\}$$

is a subgroup of $G$. (On homework.)
Recall, for all \( x, y \in G \), we have \( xy = yx \) if and only if \( xyx^{-1} = 1 \). We call the expression \( xyx^{-1} \) the conjugation of \( y \) by \( x \). For example, conjugating elements of \( D_6 \) looks like

\[
\begin{array}{c|cccccc}
  x & 1 & r & r^2 & s & sr & sr^2 \\
  \hline
  1 & 1 & r & r^2 & s & sr & sr^2 \\
  r & 1 & r & r^2 & sr & sr^2 & s \\
  r^2 & 1 & r & r^2 & sr^2 & s & sr \\
  s & 1 & r^2 & r & s & sr^2 & sr \\
  sr & 1 & r^2 & r & sr^2 & sr & s \\
  sr^2 & 1 & r^2 & r & sr & s & sr^2 \\
\end{array}
\]

On the other hand, the subgroups of \( D_6 \) are \( \{1\} \), \( \{1, s\} \), \( \{1, sr\} \), \( \{1, sr^2\} \), \( \{1, r, r^2\} \), and \( D_6 \).

More special subgroups

Let \( A \) be a non-empty subset of \( G \) (not necessarily a subgroup). The centralizer of \( A \) in \( G \) is

\[
C_G(A) = \{ g \in G \mid gag^{-1} = a \text{ for all } a \in A \}.
\]

Since

\[
gag^{-1} = a \iff ga = ag
\]

this is the set of elements which commute with all \( a \) in \( A \).

If \( A = \{a\} \), we write \( C_G(\{a\}) = C_G(a) \).

**Theorem**

*For any non-empty \( A \subseteq G \), \( C_A(G) \) is a subgroup of \( G \).*