Last time: Direct products

Let

\[(G_1, \star_1), (G_2, \star_2), \ldots\]

be groups. Then the direct product of these groups is the set

\[G_1 \times G_2 \times \cdots = \{(g_1, g_2, \ldots) \mid g_i \in G\}\]

with binary operation

\[(g_1, g_2, \ldots) \star (g'_1, g'_2, \ldots) = (g_1 \star_1 g'_1, g_2 \star_2 g'_2, \ldots)\]

Once we're comfortable with the fact that different coordinates have their own binary operations, we can stop writing the stars like before:

\[(g_1, g_2, \ldots)(g'_1, g'_2, \ldots) = (g_1 g'_1, g_2 g'_2, \ldots)\]

(unless \(\star_i\) is +, in which case you should still write +).

Proposition

If \(G_1, \ldots, G_n\) are groups, their direct product is a group of order \(|G_1||G_2| \cdots |G_n|\).

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Proposition

Let \(G_1, \ldots, G_n\) be groups, and let \(G = \hat{G}_1 \times \cdots \times \hat{G}_n\). Fix \(i \in [n]\).

1. The set

\[\hat{G}_i = \{(1, \ldots, 1, \underbrace{g}_{i^{th} \text{ component}}, 1, \ldots, 1) \mid g \in G_i\} \subseteq G\]

is a normal subgroup isomorphic to \(G_i\). Identify \(G_i\) with this subgroup (i.e. just write \(G_i\) instead of \(\hat{G}_i\), even though they're slightly different a priori). Then

\[G/G_i \cong G_1 \times \cdots \times G_{i-1} \times G_{i+1} \times G_n\]

2. For each fixed \(i\) define the projection

\[\pi_i : G \to G_i \quad \text{by} \quad (g_1, \ldots, g_n) \mapsto g_i.\]

Then \(\pi_i\) is a surjective homomorphism with

\[\ker(\pi_i) = \{(g_1, \ldots, g_{i-1}, 1, g_{i+1}, \ldots, g_n) \mid g_j \in G_j\} \cong G/G_i\]

3. If \(x \in G_i\) and \(y \in G_j\) for some \(i \neq j\) then \(xy = yx\). So if \(x_i \in G_i\) for \(i = 1, \ldots, n\), then \((x_1 x_2 \cdots x_n)^\ell = x_1^\ell x_2^\ell \cdots x_n^\ell\), and so \(|x_1 \cdots x_n| = \text{lcm}(|x_1|, \ldots, |x_n|)\).
Proposition
Let \( m, n \in \mathbb{Z}^+ \).

1. \( \mathbb{Z}_m \times \mathbb{Z}_n \cong \mathbb{Z}_{mn} \) if and only if \((m, n) = 1\).
2. If \( n = p_1^{\alpha_1} \cdots p_k^{\alpha_k} \) then \( \mathbb{Z}_n \cong \mathbb{Z}_{p_1^{\alpha_1}} \times \cdots \times \mathbb{Z}_{p_k^{\alpha_k}} \).

Recall, for groups \( A, B, \) and \( C \),

\[ A \times B \cong B \times A \quad \text{and} \quad (A \times B) \times C \cong A \times (B \times C). \]

Examples:
\( \mathbb{Z}_2 \times \mathbb{Z}_3 \cong \mathbb{Z}_6 \), but \( \mathbb{Z}_6 \times \mathbb{Z}_4 \not\cong \mathbb{Z}_{24} \)

Note: \( \mathbb{Z}_6 \times \mathbb{Z}_4 \) is not cyclic!
Since \( 24 = 8 \times 3 = 2^3 \times 3 \), we have
\[ \mathbb{Z}_{24} \cong \mathbb{Z}_{2^3} \times \mathbb{Z}_3. \]

Can: break \( \mathbb{Z}_n \) into maximal \( \mathbb{Z}_{p^{\alpha}} \)'s, or merge \( \mathbb{Z}_{p^{\alpha}} \) with \( \mathbb{Z}_{q^{\beta}} \) \( (p \neq q) \).
Cannot: break \( \mathbb{Z}_{p^{\alpha}} \)'s down any further, or merge \( \mathbb{Z}_{p^{\alpha}} \) with \( \mathbb{Z}_{p^{\beta}} \).

Definition
1. A group \( G \) is finitely generated if there is a finite subset \( A \) of \( G \) such that \( G = \langle A \rangle \).
2. For each \( r \in \mathbb{Z}_{\geq 0} \), let \( \mathbb{Z}^r = \mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z} \) be the direct product of \( r \) copies of the group \( \mathbb{Z} \), where \( \mathbb{Z}^0 = 1 \). The group \( \mathbb{Z}^r \) is called the free abelian group of rank \( r \).

Theorem (Fundamental Theorem of Finitely Generated Abelian Groups)
Let \( G \) be a finitely generated abelian group. Then

1. \( G \cong \mathbb{Z}^r \times \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_s} \)
   for some integers \( r, n_1, \ldots, n_s \) such that \( r \geq 0 \) and
   \[ 2 \leq n_s | \cdots | n_2 | n_1. \]
2. The expression in (1) is unique.

(Presented without proof).