A comment on our proofs

In proving the 3rd iso thm, given by

\[ \text{if } H, K \leq G, \text{ then } (G/H)/(K/H) \cong G/K, \]

we studied the map

\[ \varphi : G/H \to G/K \quad \text{defined by} \quad gH \to gK. \]

The first step was to show this was well-defined (independent of choice of representative in \( gH \)).

Another way: \( \varphi \) is basically the map

\[ \Phi : G \to G/K \quad \text{defined by} \quad g \to gK, \]

except split into two parts by

\[ G \xrightarrow{\pi} G/H \xrightarrow{\varphi} G/K \]

\[ \Phi \]

General principle: homomorphisms on quotient groups

Any map \( \varphi : G/N \to H \) is a map on cosets \( gN \), so the image is determined by \( g \) itself. Essentially, \( \varphi \) pulls back to a map

\[ \Phi : G \to H \]

where \( \Phi(g) = \varphi(gN) \).

Then \( \varphi \) is well-defined

if and only if \( \varphi(gN) = \varphi(g'N) \) for all \( g' \in gN \)

if and only if \( \Phi(g) = \Phi(g') \) for all \( g'N \in gN \)

if and only if \( N \leq \ker(\Phi) \).

So to define a homomorphism \( \varphi : G/N \to H \), you can instead define a homomorphism \( \Phi : G \to H \) and check that \( N \leq \ker(\Phi) \).

In this case, we say \( \Phi \) factors through \( N \) and \( \varphi \) is the induced homomorphism on \( G/N \). Pictorially, we say the following diagram commutes, meaning that following either path from \( G \) to \( H \) gives the same image:

\[ G \xrightarrow{\pi} G/N \]

\[ \Phi \]

\[ \varphi \]

\[ H \]
Fourth (lattice) isomorphism theorem

Theorem
Let \( N \trianglelefteq G \).
There natural projection \( \pi : G \to G/N \) gives a bijection

\[
\{ A \mid N \trianglelefteq A \trianglelefteq G \} \longleftrightarrow \{ \bar{A} \mid \bar{A} \trianglelefteq G/N \}
\]

where \( \bar{A} = \pi(A) = A/N \).

For all \( N \trianglelefteq A, B \trianglelefteq G \), this bijection additionally satisfies

1. \( A \trianglelefteq G \) if and only if \( \bar{A} \trianglelefteq \bar{G} \),
2. \( A \trianglelefteq B \) if and only if \( \bar{A} \trianglelefteq \bar{B} \),
3. if \( A \trianglelefteq B \), then \( |B : A| = |\bar{B} : \bar{A}| \),
4. \( \langle A, B \rangle = \langle \bar{A}, \bar{B} \rangle \).
Lattice of $D_{16}$
The Hölder program, a story

**Goal:** Classify, up to isomorphism, all finite groups.

**Lattice isomorphism theorem:**

"$G/N$ is the group those structure describes the structure of $G$ above $N$"

This gives us an idea of how to use induction (on size) to simplify the study of finite groups.

“If $N \trianglelefteq G$, then the structure of $G$ is equivalent to

1. the structure of $G/N$,
2. the structure of $N$, and
3. how $G$ is ‘built’ from $N$ and $G/N$"

**Base case:** We say a group $G$ is **simple** if $|G| > 1$ and the only normal subgroups of $G$ are $G$ and 1.

**Hölder program:**

1. Classify all finite simple groups.
2. Find all ways of “putting simple groups together”.

Classifying finite simple groups

**Theorem**

_Every finite simple group is isomorphic to one of_

1. a group in one of 18 (infinite) families, or
2. one of 26 “sporadic” groups.

For example,

$$A = \{Z_p \mid p \text{ prime}\}$$

is one of the 18 infinite families of finite simple groups. Namely, if $p$ is prime, then $Z_p$ is simple (by Lagrange); and if $p \neq p'$, then $Z_p \ncong Z_{p'}$.

The sort of theorem that went into the classification is as follows:

**Theorem (Feit-Thompson).** _If $G$ is simple and of odd order, then $G \cong Z_p$ for some prime $p$._

(The proof of this theorem took about 255 pages of sophisticated and technical mathematics. We won’t do it.)
Classifying finite simple groups

Theorem
Every finite simple group is isomorphic to one of
1. a group in one of 18 (infinite) families, or
2. one of 26 “sporadic” groups.

Another example of one of the 18 families is
\( B = \{ \text{SL}_n(\mathbb{F})/Z(\text{SL}_n(\mathbb{F})) \mid n \in \mathbb{Z}_{\geq 2}, \mathbb{F} \text{ is a finite field} \} \).

Namely,
\[
\text{SL}_n(\mathbb{F}) = \{ M \in \text{GL}_n(\mathbb{F}) \mid \det(M) = 1 \}
\]
is finite if \( \mathbb{F} \) is finite. But \( \text{SL}_n(\mathbb{F}) \) is not simple, since \( Z(G) \) is always normal in \( G \), and
\[
Z(\text{SL}_n(\mathbb{F})) = \left\{ \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ & \lambda_2 & \cdots & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} \mid \lambda_i \in \mathbb{F}, \ \lambda_1 \lambda_2 \cdots \lambda_n = 1 \right\} \neq 1
\]
if \( |\mathbb{F}| > 1 \). It turns out, though, that \( \text{SL}_n(\mathbb{F})/Z(\text{SL}_n(\mathbb{F})) \) is simple.

And the groups in \( B \) are distinct for distinct \( n \) and \( \mathbb{F} \).

“Putting simple groups together”

Let \( A \) and \( B \) groups. “Putting \( A \) and \( B \) together” means finding a group \( G \) with normal subgroups \( N \) such that \( G/N \cong A \) and \( N \cong B \).

Example: \( B \cong 1 \times B \cong A \times B \) and \( (A \times B)/(1 \times B) \cong A \).

Example: The lattices for \( Q_8 \) and \( D_8 \) are

Highlighted the sublattices corresponding to \( Q_8/\langle -1 \rangle \) and \( D_8/\langle r^2 \rangle \) shows that they have the same lattice. In fact,
\[
Q_8/\langle -1 \rangle \cong Z_2 \times Z_2 \cong D_8/\langle r^2 \rangle.
\]
Further, \( \langle -1 \rangle \cong \langle r^2 \rangle \). So
\[
G/N \cong G'/N' \text{ and } N \cong N' \text{ does not imply } G \cong G'!
\]

Point: There’s more than one way to put \( A \) and \( B \) together.