Warm-up

(1) Let \( H = \{1, r^2, r^4\} \subset D_{12} \).
   (a) Check that \( H \trianglelefteq D_{12} \).
       (Check both that it’s closed, and therefore a subgroup, and that
        \( xHx^{-1} = H \) for both generators \( x = r \) and \( s \).)
   (b) List the elements of \( D_{12}/H \).
       (Before you start, how many are there?)
   (c) Show \( D_{12}/H \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \).

(2) Let \( G = \mathbb{Z}_3 \times \mathbb{Z}_3 \) and let \( H = \langle (x, 1) \rangle \), where \( \mathbb{Z}_3 = \langle x \rangle \).
   (a) Briefly, how do we know \( H \trianglelefteq G \)?
   (b) Give a multiplication table for \( G/H \).
   (c) Show that \( G \) is not cyclic, but \( G/H \) is cyclic.

Big theorems from last time

For a subgroup \( H \trianglelefteq G \), define the set
\[
G/H = \{gH \mid g \in G\}.
\]

Proposition

A subgroup \( N \) of \( G \) is normal if and only if it is the kernel of some homomorphism.

Proof. First, we showed kernels were normal, and \( G/\ker \) is a group. Then we showed that \( G/N \) is a group if and only if \( N \trianglelefteq G \), and if \( N \trianglelefteq G \) then
\[
\pi : G \to G/N \quad \text{defined by} \quad g \mapsto gN
\]
is a bijective homomorphism, with kernel \( N \).

Theorem (First isomorphism theorem)

If \( \varphi : G \to H \) is a homomorphism of groups, then
\[
G/\ker(\varphi) \cong \varphi(G).
\]

Punchline: The study of the isomorphism classes of homomorphic images of \( G \) is “the same” as the study of the normal subgroups of \( G \).
Some consequences of coset results

**Theorem (Lagrange)**
Let $G$ be a finite group. If $H \leq G$, then $|H|$ divides $|G|$, and the number of cosets of $H$ is equal to $|G|/|H|$.

**Corollary**
Let $G$ be a finite group.

1. If $N \trianglelefteq G$, then $|G/N| = |G|/|N|$.
2. The order of any element $g \in G$ divides $|G|$.
3. If $G$ has prime order, then $G$ is cyclic.
4. If $A, B \leq G$, and $AB = \{ab \mid a \in A, b \in B\}$ then
   
   $$|AB| = |A||B|/|A \cap B|.$$  

**Definition**
If $G$ is a (possibly infinite) group and $H \leq G$, the number of left cosets of $H$ in $G$ is the **index** of $H$ in $G$, written $|G : H|$.
Building bigger subgroups
We showed in the exam that if $A, B \leq G$ then
\[ A \cap B \leq G \quad \text{but} \quad A \cup B \leq G \text{ iff } A \cup B = A \text{ or } B. \]
The way to build bigger subgroups: Let
\[ AB = \{ab \mid a \in A, b \in B\}. \]

Proposition
$AB \leq G$ if and only if $AB = BA$.

Isomorphism theorems, a preview

Let $G$ be a group.
1. If $\varphi : G \rightarrow H$ is a homomorphism of groups, then $\ker(\varphi) \leq G$ and
   \[ G/\ker(\varphi) \cong \varphi(G). \]
2. Let $A, B \leq G$ and assume $A \leq N_G(B)$. Then
   \[ AB/B \cong A/(A \cap B) \]
   (with appropriate statements about normality).
3. Let $A, B \leq G$ with $A \leq B$. Then $B/A \leq G/A$ and
   \[ (G/A)/(B/A) \cong (G/B). \]
4. Every subgroup of $G/N$ comes from projecting a subgroup of $G$, and the containment, generation, normality, and index information pass through via $\pi$ the way you want them to.
Second “diamond” isomorphism theorem

We just showed that for $A, B \trianglelefteq G$, and $AB = \{ab \mid a \in A, b \in B\}$, we have

$AB \trianglelefteq G$ if and only if $AB = BA$.

We also showed (while calculating $|AB|$) that for $a, a' \in A$,

$aB = a'B$ if and only if $a(A \cap B) = a'(A \cap B)$.

Theorem

Suppose $A \trianglelefteq N_G(B)$ (we say $A$ normalizes $B$)

1. Then $AB \trianglelefteq G$.
   (In general, if $B \trianglelefteq G$, then $AB \trianglelefteq G$ for any $A \trianglelefteq G$.)

2. Additionally, $B \trianglelefteq AB$, $A \cap B \trianglelefteq A$ and

   $AB/B \cong A/(A \cap B)$. 

\[
\begin{tikzpicture}
  \node (G) at (0,0) {$G$};
  \node (AB) at (1,-1) {$AB$};
  \node (A) at (0,-1.5) {$A$};
  \node (B) at (1,-1.5) {$B$};
  \node (A_cap_B) at (0.5,-2) {$A \cap B$};
  \draw (G) -- (AB);
  \draw (AB) -- (A);
  \draw (AB) -- (B);
  \draw (G) -- (A_cap_B);
  \draw (A) -- (A_cap_B);
  \draw (B) -- (A_cap_B);
  \draw (G) -- (1,0);
\end{tikzpicture}
\]
Third isomorphism theorem

**Theorem**

Let $A, B \leq G$ with $A \leq B$. Then

$$A \leq B, \quad B/A \cong G/A,$$

and

$$(G/A)/(B/A) \cong G/B.$$ 

**Example:**

$$(\mathbb{Z}/6\mathbb{Z})/(2\mathbb{Z}/6\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}.$$