Last time:

Recall that the fibers of a map $\varphi : X \to Y$ are the sets in $\varphi^{-1}(y) \subseteq X$ which all map to the same element $y \in Y$.

**Example 1:** The fibers of the homomorphism

$$
\varphi : \mathbb{Z} \to \mathbb{Z}/4\mathbb{Z}
$$

$$
z \mapsto \bar{z}.
$$

are

$$
4\mathbb{Z} = \{4z \mid z \in \mathbb{Z}\} = \{\ldots, -8, -4, 0, 4, 8, \ldots\} \mapsto 0
$$

$$
4\mathbb{Z} + 1 = \{4z + 1 \mid z \in \mathbb{Z}\} = \{\ldots, -7, -3, 1, 5, 9, \ldots\} \mapsto 1
$$

$$
4\mathbb{Z} + 2 = \{4z + 2 \mid z \in \mathbb{Z}\} = \{\ldots, -6, -2, 2, 6, 10, \ldots\} \mapsto 2
$$

$$
4\mathbb{Z} + 3 = \{4z + 3 \mid z \in \mathbb{Z}\} = \{\ldots, -5, -1, 3, 7, 11, \ldots\} \mapsto 3
$$

Last time:

**Example 2:** The map

$$
\varphi : D_{12} \to S_3 \text{ defined by}
$$

$$
s \mapsto (12)
$$

$$
r \mapsto (123)
$$

extends to a homomorphism. Its fibers are

$$
K = K = \ker(\varphi) = \{1, r^3\} \mapsto 1
$$

$$
Ks = \{ks \mid k \in K\} = \{s, r^3s\} \mapsto (12)
$$

$$
Krs = \{krs \mid k \in K\} = \{rs, r^4s\} \mapsto (13)
$$

$$
Kr^2s = \{kr^2s \mid k \in K\} = \{r^2s, r^5s\} \mapsto (23)
$$

$$
Kr = \{kr \mid k \in K\} = \{r, r^4\} \mapsto (123)
$$

$$
Kr^2 = \{kr^2 \mid k \in K\} = \{r^2, r^5\} \mapsto (132)\]
Last time:
Let $K \trianglelefteq G$. Then for $g \in G$, we call the sets
$$gK = \{gk \mid k \in K\} \quad \text{and} \quad Kg = \{kg \mid k \in K\}$$
the left and right coset of $K$ (corresponding to $g$). And element of a coset is called a representative of the coset.

Theorem
Let $\varphi : G \to H$ be a surjective homomorphism of groups with kernel $K$. For each $h \in H$, let
$$X_h = \varphi^{-1}(h) = \{g \in G \mid \varphi(g) = h\}.$$

1. Then
$$x \in X_a \text{ and } y \in X_b \quad \text{implies} \quad xy \in X_{ab}.$$  
   In particular as subsets of $G$, $\{X_h \mid h \in H\}$ is a group under the operation $X_a \star X_b = X_{ab}$. (We call this group the quotient group $G/\ker(\varphi)$).

2. Fix some fiber $X_h$. For any $x \in X_h$,
$$X_h = xK \quad \text{and} \quad X_h = Kx.$$
“2. Fix some fiber $X_h$. For any $x \in X_h$, 
\[ X_h = xK \quad \text{and} \quad X_h = Kx. \]
This says that if $K \leq G$ is the kernel of some homomorphism, then 
\[ xK = Kx \quad \text{for all } x \in G, \quad \text{and} \quad Kx = Ky \quad \text{for all } y \in Kx \]
(the sets are equal).

Caution: Kernels are special — left and right cosets, even of subgroups, aren’t always equal. For example, consider 
\[ G = S_3, \quad H = \langle (12) \rangle = \{1, (12)\}, \quad \text{and} \quad x = (23). \]
These give 
\[ xH = \{(23)1, (23)(12)\} = \{(23), (132)\} \]
and 
\[ Hx = \{1(23), (12)(23)\} = \{(23), (123)\}. \]

“1. Then 
\[ x \in X_a \quad \text{and} \quad y \in X_b \quad \text{implies} \quad xy \in X_{ab}. \]
In particular as subsets of $G$, $\{X_h \mid h \in H\}$ is a group under the operation $X_a \ast X_b = X_{ab}$.”
This says that if $K$ is the kernel of some homomorphism of a group $G$, then the set 
\[ G/K := \{xK \mid x \in G\} \]
is a group under the multiplication 
\[ xK \ast yK = (xy)K. \]
Further, if $K$ is, specifically, the kernel of the homomorphism $\varphi : G \to H$, then the map 
\[ G/K \to \text{img}(\varphi) \leq H \quad \text{defined by} \quad xK \mapsto \varphi(x) \]
is a bijective homomorphism, so $G/\ker(\varphi) \cong \text{img}(\varphi)$. 
(This is called the 1st isomorphism theorem)
Skip the homomorphism

When do the set of cosets of a set \( A \subseteq G \) form a group? i.e. when is the multiplication

\[
xA \ast yA = (xy)A
\]

well-defined? No hope if \( A \) is not a group!

**Proposition**

Let \( H \trianglelefteq G \).

1. The left cosets of \( H \) partition the elements of \( G \).
2. \( xH = yH \) if and only if \( y^{-1}x \in H \).

For 1, consider the equivalence relation \( x \sim y \) if \( x \in yH \ldots \)

**Proposition**

Let \( H \trianglelefteq G \).

1. The operation \( \ast \) in (*) is well defined if and only if \( gxg^{-1} \in H \) for all \( x \in H \) and \( g \in G \).
2. If \( \ast \) is well-defined, then \( G/H = \{ gH \mid g \in G \} \) forms a group with \( 1 = 1H \) and \( (gH)^{-1} = g^{-1}H \).
Definition
A subgroup $N \leq G$ is **normal** if $gNg^{-1} = N$ for all $g \in G$, i.e. when $G/N$ is a well-defined group. Write $N \trianglelefteq G$.

In summary, let $N \leq G$. Then the following are equivalent:

1. $N$ is normal in $G$
2. $N_G(N) = G$
3. $gN = Ng$ for all $g \in G$
4. the operation on left cosets given by $xN \ast yN = (xy)N$ is well-defined
5. $gN g^{-1} \subseteq N$ for all $g \in G$
Example: Letting $D_8$ act on itself by conjugation ($g \cdot a = gag^{-1}$) yields the following table:

\[
\begin{array}{c|cccccccc}
g h g^{-1} & 1 & r & r^2 & r^3 & s & sr & sr^2 & sr^3 \\
\hline
1 & 1 & r & r^2 & r^3 & s & sr & sr^2 & sr^3 \\
r & 1 & r & r^2 & r^3 & sr^2 & sr^3 & s & sr \\
r^2 & 1 & r & r^2 & r^3 & s & sr & sr^2 & sr^3 \\
r^3 & 1 & r & r^2 & r^3 & sr^2 & sr^3 & s & sr \\
s & 1 & r^3 & r^2 & r & s & sr^3 & sr^2 & sr \\
sr & 1 & r^3 & r^2 & r & sr^2 & sr & s & sr^3 \\
sr^2 & 1 & r^3 & r^2 & r & s & sr^3 & sr^2 & sr \\
sr^3 & 1 & r^3 & r^2 & r & sr^2 & sr & s & sr^3 \\
\end{array}
\]

What are the normal subgroups of $D_8$?

Note: "subgroup" is transitive but "normal" is not!! For example, \(\{1, s\} \trianglelefteq \{1, r^2, s, sr^2\}\) and \(\{1, r^2, s, sr^2\} \trianglelefteq D_8\), but \(\{1, s\} \ntrianglelefteq D_8\).
More examples

1. $G$ and $1 = \{1_G\}$ are always normal in $G$, with

   $$G/1 \cong G \quad \text{and} \quad G/G \cong 1.$$ 

2. All subgroups of the center $Z(G)$ of a group are normal in $G$.

3. All subgroups of abelian groups are normal.

4. Quotients of abelian groups are abelian and quotients of cyclic groups are cyclic.
You try: Recall $Q_8$ is generated by $-1, i, j, k$, with relations

\[
(-1)^2 = 1, \quad i^2 = j^2 = k^2 = -1, \\
i \cdot j = k = -j \cdot i, \quad j \cdot k = i = -k \cdot j, \quad k \cdot i = j = -i \cdot k.
\]
Let $H = \langle -1 \rangle$.

(1) Show that $H \cong Q_8$, and conclude $Q_8/H$ is a group.

(2) List the elements of $Q_8/H$ (pick one rep. per coset).

(3) Give a multiplication table for $Q_8/H$, and give a presentation for $Q_8/H$.

(4) Show $Q_8/H \cong Z_2 \times Z_2$ (where $Z_2 = \langle x \mid x^2 = 1 \rangle$).