1. Define “cardinality”: Two sets $X$ and $Y$ have the same cardinality if... 

[Hint: “...they have the same size” is not the right answer.]

there is a bijection

$\phi: X \rightarrow Y$.

2. Define “surjective function”: A function $f : X \rightarrow Y$ is surjective if...

$f(Y) = Y$

(for every $y \in Y$, there is some $x \in X$ such that $f(x) = y$.)

3. Give an example of a function that is injective but not bijective.

$f: \{1, 3\} \rightarrow \{1, 2, 3\}$ defined by

$\begin{align*}
    1 & \mapsto 1, \\
    3 & \mapsto 2.
\end{align*}$

4. Give a closed formula for the sequence defined recursively by $a_n = 3a_{n-1}$ and $a_0 = 5$.

This is a geometric sequence:

\[ a_n = r^n \cdot a_0 \quad \text{where} \quad r = 3, \quad \text{ie} \]

\[ a_n = 3^n \cdot 5. \]
5. Outline a proof by induction that \( \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \).

- Define \( P(n) : \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \).

- Base case: \( P(1) \) is

\[
\sum_{i=1}^{1} i = 1 = \frac{1 \cdot 2}{2} \quad \checkmark
\]

- Goal: fix \( n \geq 1 \) and assume \( P(n) \), then show \( P(n+1) \), which is

\[
\sum_{i=1}^{n+1} i = \frac{(n+1)(n+2)}{2}.
\]

- Inductive step: Fix \( n \geq 1 \), and assume

\[
\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \quad \text{(for that } n \text{)}.
\]

Then

\[
\sum_{i=1}^{n+1} i = \frac{n(n+1)}{2} + (n+1)
\]

\[
= \frac{n(n+1)}{2} + (n+1) \quad \text{(by the induction hypothesis)}
\]

\[
= \frac{n^2 + n + 2n + 2}{2} = \frac{(n+1)(n+2)}{2}.
\]

- Conclusion: Since \( P(1) \) is true and \( P(n) \) implies \( P(n+1) \) for \( n \geq 1 \), we have \( P(n) \) holds for all \( n \geq 1 \).