Math 365 – Quiz 2 – Wednesday 2/20/19

Solutions
Name

1. Define "cardinality": *Two sets X and Y have the same cardinality if...*
[Hint: "...they have the same size" is not the right answer.]

there is a bijection

$f: X \to Y$.

2. Define "surjective function": *A function $f : X \to Y$ is surjective if...*

$f(X) = Y$

(for every $y \in Y$, there is some $x \in X$ such that $f(x) = y$.)

3. Give an example of a function that is injective but not bijective.

$f: \{1\} \to \{1, 2\}$ defined by

$1 \mapsto 1$.

4. Give a closed formula for the sequence defined recursively by $a_n = 3a_{n-1}$ and $a_0 = 5$.

This is a geometric sequence:

$a_n = r^n \cdot a_0$ where $r = 3$, ie

$\boxed{a_n = 3^n \cdot 5.}$

5. Outline a proof by induction that $\sum_{i=1}^{n} i = n(n+1)/2$.

- Define $P(n)$: $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

- Base case: $P(1)$ is

$$\sum_{i=1}^{1} i = 1 = \frac{1 \cdot 2}{2}. \checkmark$$

- Goal: fix $n \geq 1$ and assum $P(n)$,
  then show $P(n+1)$, which is

$$\sum_{i=1}^{n+1} i = \frac{(n+1)(n+2)}{2}.$$

- Inductive step: Fix $n \geq 1$, and assume

$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ (for that n).

Then

$$\sum_{i=1}^{n+1} i = \underbrace{1 + 2 + \dots + n}_{\sum_{i=1}^{n} i} + (n+1)$$

$$= \frac{n(n+1)}{2} + (n+1) \quad \text{(by the inductive hypothesis)}$$

$$= \frac{n^2 + n + 2n + 2}{2} = \frac{(n+1)(n+2)}{2}.$$

- Conclusion Since $P(1)$ is true and $P(n)$ implies $P(n+1)$ for $n \geq 1$, we have $P(n)$ holds for all $n \geq 1$.