SOLUTIONS
Homework 8
Due 4/3/19

Exercise 37. Recall that the generating function for a sequence $\{a_0, a_1, a_2, \ldots, \}$ is

$$\sum_{k=0}^{\infty} a_k x^k.$$

(Considered "formally", i.e. without consideration of convergence.)

Give the generating functions, in terms of their series and in their simplified (closed) form, for each of the following sequences. In many cases, you'll use your answers from Exercise 36 (from last time).

(a) $a_n = n + 1$ for $n = 0, 1, 2, \ldots$.

Answer.

$$G(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} (n+1)x^n = \frac{1}{(1-x)^2}.$$

....

(b) $a_n = 3^n$, for $n = 0, 1, \ldots$.

Answer.

$$G(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a^n x^n = \frac{1}{1-ax}.$$

....

(c) $a_n = \begin{cases} 1 & n = 3k \text{ for some } k \in \mathbb{Z}, \\ 0, & n \neq 3k \text{ for all } k \in \mathbb{Z}, \end{cases}$ for $n = 0, 1, \ldots$.
(i.e. a_n is 1 if n is a multiple of 3, and is 0 otherwise).

Answer.

$$G(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} x^{3n} = \frac{1}{1-x^3}.$$

....

(d) $a_n = 1/n$ for $n = 1, 2, \ldots$.

Answer.

$$G(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=1}^{\infty} \frac{1}{n} x^n = -\ln(1-x).$$

....

Exercise 38. For each of the following recursively defined sequences,

(i) solve using generating functions;
(ii) solve using the methods of Section 8.2 and compare to part (i);
(iii) check your answers by comparing the values you get by using your formula and by computing recursively for the first three terms of the sequence.

(a) $a_n = 2a_{n-1} + 3$, $a_0 = 1$.

Answer.

(i) *Using generating functions:*

Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$. Then

$$\begin{aligned} G(x) &= \sum_{n=0}^{\infty} a_n x^n = a_0 + \sum_{n=1}^{\infty} a_n x^n = 1 + \sum_{n=1}^{\infty} (2a_{n-1} + 3)x^n \\ &= 1 + 2\sum_{k=1}^{\infty} a_k x^{k+1} + \sum_{n=1}^{\infty} 3x^n \\ &= 1 + 2xG(x) + 3x\frac{1}{1-x}. \end{aligned}$$

so

$$G(x)(1-2x) = 1 + \frac{3x}{1-x} = \frac{1+2x}{1-x}, \qquad \text{so that} \qquad G(x) = \frac{1+2x}{(1-x)(1-2x)}.$$

Using the method of partial fractions, we want to rewrite

$$G(x) = \frac{1+2x}{(1-x)(1-2x)} = \frac{A}{1-2x} + \frac{B}{1-x}.$$

This is equivalent to

$$1 + 2x = A(1-x) + B(1-2x) = (A+B) - x(A+2B)$$

So

$$1 = A + B \quad \text{and} \quad 2 = -A - 2B,$$

meaning $A = -2(1+B)$ and so $1 = -2 - 2B + B = -2 - B$. Thus $B = -3$ and $A = 4$. So

$$\begin{aligned} G(x) &= \frac{3}{(1-x)(1-2x)} = 4\frac{1}{1-2x} - 3\frac{1}{1-x} \\ &= 4\sum_{n=0}^{\infty} 2^n x^n - 3\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} (4 * 2^n - 3)x^n. \end{aligned}$$

Thus

$$\boxed{a_n = 4 * 2^n - 3.}$$

(ii) *Using the methods of Section 8.2 and compare:*

This is a linear and constant coefficient recurrence relation, but is not homogeneous. The associated homogeneous part is $h_n = 2h_{n-1}$. The associated characteristic equation is $0 = r - 2$. So the general (homogeneous) solution is

$$h_n = \alpha 2^n.$$

For the particular solution, guess $\hat{a}_n = p_0$. Plugging in, we have

$$\begin{aligned} \hat{a}_n &= p_0 \\ &= 2\hat{a}_{n-1} + 3 = 2p_0 + 3. \end{aligned}$$

Thus $p_0 = -3$, so that

$$a_n = h_n + \hat{a}_n = \alpha 2^n - 3.$$

Plugging in $a_0 = 1$, we get

$$1 = a_0 = \alpha 2^0 - 3 = \alpha - 3, \qquad \text{so that } \alpha = 4.$$

Thus

$$\boxed{a_n = 4 * 2^n - 3}$$

(iii) *Check by computing recursively for the first three terms of the sequence:*
We have

$$\begin{aligned} a_0 &= 1 = 4 * 2^0 - 3; & \checkmark \\ a_1 &= 2a_0 + 3 = 5 = 4 * 2 - 3; & \checkmark \\ a_2 &= 2a_1 + 3 = 13 = 4 * 2^2 - 3. & \checkmark \end{aligned}$$

. . . .

(b) $a_n = 3a_{n-1} + 4^{n-1}$.

Answer.

(i) *Using generating functions:*
Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$. Then

$$\begin{aligned} G(x) &= \sum_{n=0}^{\infty} a_n x^n = a_0 + \sum_{n=1}^{\infty} a_n x^n \\ &= a_0 + \sum_{n=1}^{\infty} (3a_{n-1} + 4^{n-1}) x^n = a_0 + 3 \sum_{k=0} a_k x^{k+1} + \sum_{k=0} 4^k x^{k+1} \\ &= a_0 + 3xG(x) + \frac{x}{1-4x}, \end{aligned}$$

so that

$$G(x)(1-3x) = a_0 + \frac{x}{1-4x} = \frac{a_0 + (1-4a_0)x}{1-4x},$$

and hence

$$G(x) = \frac{a_0 + (1-4a_0)x}{(1-4x)(1-3x)}.$$

Using the method of partial fractions, we want to rewrite

$$G(x) = \frac{a_0 + (1-4a_0)x}{(1-4x)(1-3x)} = \frac{A}{1-4x} + \frac{B}{1-3x}.$$

This is equivalent to

$$a_0 + (1-4a_0)x = A(1-3x) + B(1-4x) = (A+B) - x(3A+4B)$$

So

$$a_0 = A + B \quad \text{and} \quad 1 - 4a_0 = -3A - 4B,$$

meaning $A = a_0 - B$ and so $1 - 4a_0 = -3a_0 + 3B - 4B = -3a_0 - B$. Thus $B = a_0 - 1$ and $A = 1$. So

$$\begin{aligned} G(x) &= \frac{a_0 + (1-4a_0)x}{(1-4x)(1-3x)} = \frac{1}{1-4x} + \frac{a_0 - 1}{1-3x} \\ &= \sum_{n=0}^{\infty} 4^n x^n + (a_0 - 1) \sum_{n=0}^{\infty} 3^n x^n = \sum_{n=0}^{\infty} (4^n + (a_0-1)3^n) x^n. \end{aligned}$$

Thus

$$\boxed{a_n = 4^n + (a_0 - 1)3^n.}$$

(ii) *Using the methods of Section 8.2 and compare:*

This is a linear and constant coefficient recurrence relation, but is not homogeneous. The associated homogeneous part is $h_n = 3h_{n-1}$. The associated characteristic equation is $0 = r - 3$. So the general solution is

$$n_n = \alpha 3^n.$$

For the particular solution, guess

$$\hat{a}_n = p_0 4^n.$$

Plugging in, we have

$$\begin{aligned} \hat{a}_n &= p_0 4^n \\ &= 3\hat{a}_{n-1} + 4^{n-1} = 3p_0 4^{n-1} + 4^{n-1}. \end{aligned}$$

So $4p_0 = 3p_0 + 1$, so that $p_0 = 1$ Thus

$$a_n = h_n + \hat{a}_n = \alpha 3^n + 4^n.$$

Rewriting in terms of a_0, we get

$$a_0 = \alpha 3^0 + 4^0 = \alpha + 1.$$

So $\alpha = a_0 - 1$, and hence

$$\boxed{a_n = 4^n + (a_0 - 1)3^n.}$$

(iii) *Check by computing recursively for the first three terms of the sequence:*

We have

$$\begin{aligned} a_0 &= 4^0 + (a_0 - 1)3^0; && \checkmark \\ a_1 &= 3a_0 + 4^0 = 3a_0 + 1 = 4^1 + (a_0 - 1)3^1; && \checkmark \\ a_2 &= 3a_1 + 4^1 = 3(3a_0 + 1) + 4 = 9a_0 + 7 = 4^2 + (a_0 - 1)3^2. && \checkmark \end{aligned}$$

. . . .

Exercise 39.

(a) Consider the integer partitions of 4.

(i) Write a corresponding generating function, wherein the coefficient of x^4 is the number of integer partitions of 4.

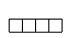
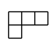
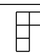
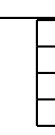
Answer.

$$\underbrace{(1+x+x^2+x^3+x^4)}_{\text{parts of length 1}}\ \underbrace{(1+x^2+x^4)}_{\text{parts of length 1}}\ \underbrace{(1+x^3)}_{\text{pts of lgth 3}}\ \underbrace{(1+x^4)}_{\text{pts of lgth 4}}$$

. . . .

(ii) Make explicit the correspondence between the monomials in the expansion of your generating function and the integer partitions of 4 (like we did for the partitions of 5 in class).

Answer.

				
$1*1*1*x^4$	$x*1*x^3*1$	$1*(x^2)^2*1*1$	$(x)^2*x^2*1*1$	$(x)^4*1*1*1$

. . . .

(b) Describe a combinatorial problem that is solved by calculating the coefficient of x^6 for the following generating functions.

(i) $(1+x+x^2+x^3)^4$

Answer. Example: The number of ways to give 3 identical cookies to four children will be the coefficient on x^3.

. . . .

(ii) $(1+x+x^2+\cdots+)(1+x^2+x^4+x^6)(1+x^3+x^6)$

Answer. Example: The coefficient of x^6 will be the number of partitions of 6 with parts of length at most 3.

. . . .

(iii) $(x+x^2+x^3)^5$

Answer. The coefficient on x^8 will be the number of ways of distributing 8 cookies to 5 children so that every child gets at most 3 and at least 1 cookie.

. . . .

(c) For each of the following questions, give the corresponding generating function and determine k such that the coefficient of x^k is the answer the question.

(i) How many different ways can 12 identical action figures can be given to five children so that each child receives at most three action figures?

Answer. The coefficient of x^{12} in

$$(1+x+x^2+x^3)^5.$$

. . . .

(ii) How many different ways 10 identical balloons can be given to four children if each child receives at least two balloons?

Answer. The coefficient of x^{10} in

$$(x^2 + x^3 + x^4 + x^5 + \cdots + x^{10})^4.$$

(Note that this is the same as the coefficient of x^{10} in $(x^2 + x^3 + x^4)^4$.)

(iii) How many different ways are there to choose a dozen bagels from three varieties–plain, onion, and raisin–if at least two bagels of each kind but no more than three plain bagels are chosen?

Answer. The coefficient of x^{12} in

$$(x^2 + x^3)(x^2 + x^3 + \cdots + x^{12})^2.$$

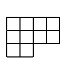
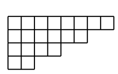
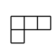
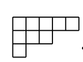
(iv) How many ways can you make change for \$100 using \$1 bills, \$5 bills, \$10 bills, and \$20 bills?

Answer. The coefficient of x^{100} in

$$(1 + x + x^2 + \cdots + x^{100})(1 + x^5 + x^{10} + \cdots + x^{100})(1 + x^{10} + x^{20} + \cdots + x^{100})$$
$$(1 + x^{10} + x^{20} + \cdots + x^{100})(1 + x^{20} + x^{40} + \cdots + x^{100}).$$

Exercise 40. (Generating functions for partitions)

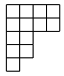
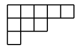
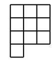
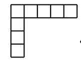
(a) Write the generating function for partitions with even-sized parts, i.e.

 or  but not  or .

Answer.

$$\sum_{n=0}^{\infty} p^{(\text{even})}(n) x^n = \prod_{k=1}^{\infty} \frac{1}{1 - x^{2k}}$$

(b) Write the generating function for partitions with no more than two parts of each size, i.e.

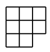
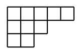
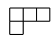
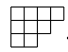
 or  but not  or .

Answer.

$$\sum_{n=0}^{\infty} p^{(\leq 2)}(n) = x^n \prod_{k=1}^{\infty} (1 + x^k + x^{2k})$$

. . . .

(c) Write the generating function for partitions parts all of prime size, i.e.

 or  but not  or .

(The prime numbers are the integers p greater than 1 that are divisible by 1 and p but no other positive integers.)

Answer.

$$\sum_{n=0}^{\infty} p^{(\text{prime})}(n) x^n = \prod_{q \text{ prime}} \frac{1}{1 - x^q}$$

. . . .