SOLUTIONS
Homework 7
Due 3/27/19

Exercise 31. For each of the following, decide if the recurrence relation is linear, homogeneous, and constant coefficient. If not, explain why it fails. If so, (i) give its degree, (ii) give its characteristic equation, and (iii) give the characteristic roots and their multiplicities.

(a) $a_n = 5a_{n-1}$

Answer. This is a linear, homogeneous, and constant coefficient recurrence relation.

(i) Degree 1.

(ii) Plug in $a_n = r^n$, assuming $r \neq 0$, and solve:

$$r^n = 5r^{n-1} \qquad \text{and so}$$
$$0 = r^n - 5r^{n-1} = r^{n-1}(r-5)$$

$$\text{Char eqn:} \qquad \boxed{0 = r - 5}$$

(iii) Characteristic roots:

roots	multiplicities
$r_1 = 5$	$m_1 = 1$

. . . .

(b) $a_n = 3$

Answer. This relation is linear and constant coefficient, but not homogeneous.

. . . .

(c) $a_n = 9a_{n-2}$

Answer. This is a linear, homogeneous, and constant coefficient recurrence relation.

(i) Degree 2.

(ii) Plug in $a_n = r^n$, assuming $r \neq 0$, and solve:

$$r^n = 9r^{n-2} \qquad \text{and so}$$
$$0 = r^n - 5r^{n-2} = r^{n-2}(r^2 - 9)$$

$$\text{Char eqn:} \qquad \boxed{0 = r^2 - 9 = (r-3)(r+3)}$$

(iii) Characteristic roots:

roots	multiplicities
$r_1 = 3$	$m_1 = 1$
$r_2 = -3$	$m_2 = 1$

. . . .

(d) $a_n = a_{n-1}^2$

Answer. This relation is not linear or homogeneous.

. . . .

(e) $a_n = -2a_{n-1} - a_{n-2}$

Answer. This is a linear, homogeneous, and constant coefficient recurrence relation.

(i) Degree 2.

(ii) Plug in $a_n = r^n$, assuming $r \neq 0$, and solve:

$$r^n = -2r^{n-1} - r^{n-2} \qquad \text{and so}$$
$$0 = r^n + 2r^{n-1} + r^{n-2} = r^{n-2}(r^2 + 2r + 1).$$

$$\text{Char eqn:} \qquad \boxed{0 = r^2 + 2r + 1 = (r+1)^2.}$$

(iii) Characteristic roots:

roots	multiplicities
$r_1 = -1$	$m_1 = 2$

. . . .

(f) $a_n = 7a_{n-1} - 6a_{n-2}$

Answer. This is a linear, homogeneous, and constant coefficient recurrence relation.

(i) Degree 2.

(ii) Plug in $a_n = r^n$, assuming $r \neq 0$, and solve:

$$r^n = 7r^{n-1} - 6r^{n-2} \qquad \text{and so}$$
$$0 = r^n - 7r^{n-1} + 6r^{n-2} = r^{n-2}(r^2 - 7r + 6).$$

$$\text{Char eqn:} \qquad \boxed{0 = r^2 - 7r + 6 = (r-1)(r-6).}$$

(iii) Characteristic roots:

roots	multiplicities
$r_1 = 1$	$m_1 = 1$
$r_2 = 6$	$m_2 = 1$

. . . .

(g) $a_n = 8a_{n-3}$

Answer. This is a linear, homogeneous, and constant coefficient recurrence relation.

(i) Degree 3.

(ii) Plug in $a_n = r^n$, assuming $r \neq 0$, and solve:

$$r^n = 8r^{n-3} \qquad \text{and so}$$
$$0 = r^n - 8r^{n-3} = r^{n-3}(r^3 - 8)$$

$$\text{Char eqn:} \qquad \boxed{0 = r^3 - 8 = (r-2)(r^2 + 2r + 4) = (r-2)(r-(1+i\sqrt{3}))(r-(1-i\sqrt{3}))}$$

(iii) Characteristic roots:

roots	multiplicities
$r_1 = 2$	$m_1 = 1$
$r_2 = 1 + i\sqrt{3}$	$m_2 = 1$
$r_3 = 1 - i\sqrt{3}$	$m_3 = 1$

. . . .

(h) $a_n = 3a_{n-1} + 4a_{n-2} - 12a_{n-3}$

Answer. This is a linear, homogeneous, and constant coefficient recurrence relation.

(i) Degree 3.

(ii) Plug in $a_n = r^n$, assuming $r \neq 0$, and solve:

$$r^n = r^{n-1} + 4r^{n-2} - 12r^{n-3} \qquad \text{and so}$$
$$0 = r^n - 3r^{n-1} - 4r^{n-2} + 12r^{n-3} = r^{n-3}(r^3 - 3r^2 - 4r + 12)$$

Char eqn: $\boxed{0 = r^3 - 3r^2 - 4r + 12 = r^2(r-3) - 4(r-3) = (r+2)(r-2)(r-3)}$

(iii) Characteristic roots:

roots	multiplicities
$r_1 = -2$	$m_1 = 1$
$r_2 = 2$	$m_2 = 1$
$r_3 = 3$	$m_3 = 1$

(i) $a_n = a_{n-1}/n$

Answer. This recurrence relation is not constant coefficient.

(j) $a_n = a_{n-1} + a_{n-2} + n + 3$

Answer. This is a linear and constant coefficient recurrence relation, but is not homogeneous.

Exercise 32. For each of the recursion relations in Exercise 31 that were linear, homogeneous, and constant coefficient, decide which have characteristic equations with k distinct roots. For those that do,

(i) write a general solution;
(ii) choose an example of initial conditions, and solve for specific α_i's; and
(iii) check your answer to the previous part by computing the first 5 terms of the sequence in two ways (recursively, and using your closed formula).

Answer.

(a) $a_n = 5a_{n-1}$:

(i) General solution:

$$a_n = \alpha_1 5^n$$

(ii) If we have initial condition

$$a_0 = 3,$$

then

$$3 = a_0 = \alpha_1 5^0 = \alpha_1$$
$$\text{so that } \alpha_1 = 3$$

So

$$a_n = 3 * 5^n.$$

(iii) Checking my answer:

$$\begin{aligned}
a_0 &= 3 = 3 * 5^0 && \checkmark \\
a_1 &= 5 * a_0 = 5 * 3 = 3 * 5^1 && \checkmark \\
a_2 &= 5 * a_1 = 5 * 5 * 3 = 3 * 5^2 && \checkmark \\
a_3 &= 5a_2 = 5 * 3 * 5^2 = 3 * 5^3 && \checkmark \\
a_4 &= 5a_3 = 5 * 3 * 5^3 = 3 * 5^4 && \checkmark
\end{aligned}$$

(c) $a_n = 9a_{n-2}$:

(i) General solution:

$$a_n = \alpha_1 3^n + \alpha_2(-3)^n.$$

(ii) If we have initial conditions

$$a_0 = 5, a_1 = -2,$$

then

$$\begin{aligned}
5 = a_0 &= \alpha_1 3^0 + \alpha_2(-3)^0 = \alpha_1 + \alpha_2 \\
&\qquad \text{so that } \alpha_2 = 5 - \alpha_1 \\
-2 &= \alpha_1 3^1 + \alpha_2(-3)^1 = 3\alpha_1 - 3(5 - \alpha_1) = 6\alpha_1 - 15.
\end{aligned}$$

So

$$\alpha_1 = 13/6 \quad \text{and } \alpha_2 = 5 - 13/6 = 17/6.$$

Thus

$$a_n = (13/6)3^n + (17/6)(-3)^n.$$

(iii) Checking my answer:

$$\begin{aligned}
a_0 &= 5 \stackrel{?}{=} (13/6)3^0 + (17/6)(-3)^0 = (13+17)/6 = 30/6 && \checkmark \\
a_1 &= -2 \stackrel{?}{=} (13/6)3^1 + (17/6)(-3)^1 = 13/2 - 17/2 = (13-17)/2 = -4/2 && \checkmark \\
a_2 &= 9a_0 = 45 \stackrel{?}{=} (13/6)3^2 + (17/6)(-3)^2 = \frac{3}{2}30 && \checkmark \\
a_3 &= 9a_1 = -18 \stackrel{?}{=} (13/6)3^3 + (17/6)(-3)^3 = \frac{3^2}{2}(-4) && \checkmark \\
a_4 &= 9a_2 = 5 * 3^4 \stackrel{?}{=} (13/6)3^4 + (17/6)(-3)^4 = \frac{3^3}{2}30 && \checkmark
\end{aligned}$$

(f) $a_n = 7a_{n-1} - 6a_{n-2}$:

(i) General solution:

$$a_n = \alpha_1 + \alpha_2 6^n$$

(ii) If we have initial conditions

$$a_0 = 3 \text{ and } a_1 = 8,$$

then

$$\begin{aligned}
3 &= a_0 = \alpha_1 + \alpha_2 \\
8 &= a_1 = \alpha_1 + \alpha_2 * 6.
\end{aligned}$$

The first equation says that $\alpha_2 = 3 - \alpha_1$. So plugging into the second equation gives

$$8 = \alpha_1 + 6(3 - \alpha_1) = 18 - 5\alpha_1.$$

So

$$\alpha_1 = -10/5 = 2; \quad \text{so that } \alpha_2 = 3 - 2 = 1.$$

So

$$a_n = 2 + 1 * 6^n = 2 + 6^n.$$

(iii) Checking my answer:

$$\begin{aligned}
a_0 &= 3 = 2 + 6^0 && \checkmark\\
a_1 &= 8 = 2 + 6^1 && \checkmark\\
a_2 &= 7 * 8 - 6 * 3 = 38 = 2 + 6^2 && \checkmark\\
a_3 &= 7 * 38 - 6 * 8 = 218 = 2 + 6^3 && \checkmark\\
a_4 &= 7 * 218 - 6 * 38 = 1298 = 2 + 6^4 && \checkmark
\end{aligned}$$

(g) $a_n = 8a_{n-3}$:

(i) General solution:

$$a_n = \alpha_1 2^n + \alpha_2(1 + i\sqrt{3})^n + \alpha_3(1 - i\sqrt{3})^n$$

(ii) If we have initial conditions

$$a_0 = 1, a_1 = 2, \text{ and } a_3 = 4,$$

then

$$\begin{aligned}
1 &= a_0 = \alpha_1 + \alpha_2 + \alpha_3\\
2 &= a_1 = \alpha_1 2 + \alpha_2(1 + i\sqrt{3}) + \alpha_3(1 - i\sqrt{3}) = 2\alpha_1 + (\alpha_2 + \alpha_3) + i\sqrt{3}(\alpha_2 - \alpha_3)\\
4 &= a_2 = \alpha_1 4 + \alpha_2(1 + i\sqrt{3})^2 + \alpha_3(1 - i\sqrt{3})^2 = \alpha_1 4 + \alpha_2(1 - i\sqrt{3}) + \alpha_3(1 + i\sqrt{3})\\
&= 4\alpha_1 + \alpha_2 + \alpha_3 + i\sqrt{3}(\alpha_3 - \alpha_2)
\end{aligned}$$

Comparing the real and the imaginary parts, the second and third equations say

$$0 = \alpha_2 - \alpha_3, \qquad \text{so } \alpha_2 = \alpha_3$$

Then the first equation says

$$1 = \alpha_1 + \alpha_2 + \alpha_3 = \alpha_1 + 2\alpha_2$$

so $\alpha_1 = 1 - 2\alpha_2$. Then the real part of the third equation says

$$2 = 2\alpha_1 + (\alpha_2 + \alpha_3) = 2(1 - 2\alpha_2) + 2\alpha_2 = 2 - 2\alpha_2.$$

So

$$\alpha_2 = 0 = \alpha_3 \quad \text{and} \quad \alpha_1 = 1 - 0 = 1.$$

So

$$a_n = 1 * 2^n + 0 * (1 + i\sqrt{3})^n + 0 * (1 - i\sqrt{3})^n = 2^n.$$

(iii) Checking my answer:

$$\begin{aligned}
a_0 &= 1 = 2^0 && \checkmark \\
a_1 &= 2 = 2^1 && \checkmark \\
a_2 &= 4 = 2^2 && \checkmark \\
a_3 &= 8a_0 = 8 * 1 = 2^3 && \checkmark \\
a_4 &= 8a_1 = 8 * 2 = 2^4 && \checkmark
\end{aligned}$$

(h) $a_n = 3a_{n-1} + 4a_{n-2} - 12a_{n-3}$:

(i) General solution:

$$a_n = \alpha_1(-2)^n + \alpha_2 2^n + \alpha_3 3^n$$

(ii) If we have initial conditions

$$a_0 = 2, a_1 = 0, a_2 = 8$$

then

$$\begin{aligned}
2 &= a_0 = \alpha_1(-2)^0 + \alpha_2 2^0 + \alpha_3 3^0 = \alpha_1 + \alpha_2 + \alpha_3 \\
0 &= a_1 = -2\alpha_1 + 2\alpha_2 + 3\alpha_3 \\
8 &= 4\alpha_1 + 4\alpha_2 + 9\alpha_3
\end{aligned}$$

The first equation says $\alpha_1 = 2 - \alpha_2 - \alpha_3$ The second and third equations tell me

$$8 = 8\alpha_2 + 15\alpha_3 \qquad 8 = 8\alpha_1 + 3\alpha_3 = -8\alpha_2 - 5\alpha_3 + 16$$

Adding these, we get $16 = 10\alpha_3 + 16$, so $\alpha_3 = 0$. Thus $8 = 8\alpha_2$, so $\alpha_2 = 1$; and $\alpha_1 = 2 - 1 - 0 = 1$. Therefore

$$a_n = (-2)^n + 2^n.$$

(iii) Checking my answer:

$$\begin{aligned}
a_0 &= 2 = (-2)^0 + (2)^0 && \checkmark \\
a_1 &= 0 = -2 + 2 && \checkmark \\
a_2 &= 8 = (-2)^2 + 2^2 && \checkmark \\
a_3 &= 3a_2 + 4a_1 - 12a_0 = 3 * 8 + 4 * 0 - 12 * 2 = 1 = 0(-2)^3 + 2^3 && \checkmark \\
a_4 &= 3a_3 + 4a_2 - 12a_1 = 0 + 4 * 8 + 0 = 32 = (-2)^4 + 2^4 && \checkmark
\end{aligned}$$

PRO TIP. Since you get to pick the initial conditions, you can reverse engineer this problem to be easy. Say you secretly want your answer to be $a_n = 2^n$. That means that $\alpha_1 = 0 = \alpha_3$ and $\alpha_2 = 1$. So use those and calculate $a_0 = 1$, $a_1 = 2$, and a_3. Then, you turn around and say "Let $a_0 = 1$, $a_0 = 2$, and $a_3 = 4$. Then..." and do the linear algebra to solve for the α's. Of course, you now know what answer you should get. Also, the "check your answer" part will be easy by design. You'll notice that I did one example above the brutal way to show the whole process, and did the others in the way that set me up for nice calculations.

. . . .

Exercise 33. For each of the recursion relations in Exercise 31 that were linear, homogeneous, and constant coefficient, but had characteristic equations with repeated roots,

(i) write a general solution;
(ii) choose an example of initial conditions, and solve for specific $\alpha_{i,j}$'s; and
(iii) check your answer to the previous part by computing the first 5 terms of the sequence in two ways (recursively, and using your closed formula).

Answer.

(e) $a_n = -2a_{n-1} - a_{n-2}$:

(i) General solution:

$$a_n = (\alpha_0 + \alpha_1 n)(-1)^n$$

(ii) If we have initial condition

$$a_0 = 0 \quad \text{and} \quad a_1 = -1,$$

then

$$\begin{aligned} 0 = a_0 &= (\alpha_0 + \alpha_1 * 0)(-1)^0 = \alpha_0 \\ &\text{so that } \alpha_0 = 0, \text{ and} \\ -1 = a_1 &= (\alpha_0 + \alpha_1 * 1)(-1)^1 = -\alpha_1 \\ &\text{so that } \alpha_1 = 1. \end{aligned}$$

So

$$a_n = n(-1)^n.$$

(iii) Checking my answer:

$$\begin{aligned} a_0 &= 0 = 0 * (-1)^0 && \checkmark \\ a_1 &= -1 = 1 * (-1)^1 && \checkmark \\ a_2 &= -2a_1 - a_0 = 2 = 2 * (-2)^2 && \checkmark \\ a_3 &= -2a_2 - a_1 = -4 + 1 = -3 = 3(-1)^3 && \checkmark \\ a_4 &= -2a_3 - a_2 = 6 - 2 = 4 = 4(-1)^2 && \checkmark \end{aligned}$$

. . . .

Exercise 34. Adapt the proof of Theorem 1 for $k = 2$ to prove Theorem 2 for $k = 2$. Namely, show that if

$$a_n = c_1 a_{n-1} + c_2 a_{n-2}$$

has a characteristic equation with a repeated root r_0, then $a_n = \alpha r_0^n + \beta n r_0^n$ is the general solution.

Outline/hints:

- Establish that the characteristic equation is $r^2 - c_1 r - c_2 = 0$.
- Justify that if r_0 is the only root of this equation, then actually $c_1 = 2r_0$ and $c_2 = -r_0^2$.
- Use a similar computation as in class to show that $a_n = \alpha r_0^n + \beta n r_0^n$ is a solution to the recurrence for any constants α and β.
- Use a similar computation as in class to show that if $\{a_n\}_{n\in\mathbb{N}}$ is a solution, then it must be of the form $a_n = \alpha r_0^n + \beta n r_0^n$ (i.e. there are some α and β that match your solution).

Exercise 35. Consider the recurrence relation

$$a_n = 8a_{n-2} - 16a_{n-4} + F(n).$$

(a) Write the associated homogeneous recursion relation and solve for its general solution $\{h_n\}$.

Answer. The associated homogeneous is $h_n = 8h_{n-2} - 16h_{n-4}$, which has characteristic equation

$$0 = r^4 - 8r^2 + 16 = (r-2)^2(r+2)^2$$

So the general solution is

$$h_n = (\alpha_0 + \alpha_1 n)2^n + (\beta_0 + \beta_1 n)(-2)^n$$

. . . .

(b) For each of the following $F(n)$, write the general form for the particular solution (<u>don't solve</u> for the unknowns).

Note: The important pieces of information from part (a) was that 2 and -2 are characteristic roots, both with multiplicity 2. So whenever 2^n or $(-2)^n$ appears in $F(n)$, I'll need to correct my guess by a factor of n^2. I have highlighted those in blue.

(i) $F(n) = n^3$ — *Answer:* $\hat{a}_n = c_0 + c_1 n + c_2 n^2 + c_3 n^3$.

(ii) $F(n) = n2^n$ — *Answer:* $\hat{a}_n = n^2(c_0 + c_1 n)2^n$

(iii) $F(n) = (n^2 - 2)(-2)^n$ — *Answer:* $\hat{a}_n = n^2(c_0 + c_1 n + c_2 n^2)(-2)^n$

(iv) $F(n) = 2$ — *Answer:* $\hat{a}_n = c_0$

(v) $F(n) = (-2)^n$ — *Answer:* $\hat{a}_n = c_0 n^2(-2)^n$

(vi) $F(n) = n^2 4^n$ — *Answer:* $\hat{a}_n = (c_0 + c_1 n + c_2 n^2)(4)^n$

(vii) $F(n) = n^4 2^n$ — *Answer:* $\hat{a}_n = n^2(c_0 + c_1 n + c_2 n^2 + c_3 n^3 + c_4 n^4)2^n$

(c) Find the general solution to $a_n = 8a_{n-2} - 16a_{n-4} + (-2)^n$.

Answer. Guess $\hat{a}_n = c_0 n^2(-2)^n$ and plug in:

$$\begin{aligned}
\hat{a}_n &= c_0 n^2(-2)^n \\
&= 8\hat{a}_{n-2} - 16\hat{a}_{n-4} + (-2)^n \\
&= 8c_0(n-2)^2(-2)^{n-2} - 16c_0(n-4)^2(-2)^{n-4} + (-2)^n \\
&= 8c_0(-2)^n\left(\left(-\frac{1}{2}\right)^2(n^2-4n+4) - 2\left(-\frac{1}{2}\right)^4(n^2-8n+16)\right) + (-2)^n \\
&= 8c_0(-2)^n\left(\frac{1}{4}(n^2-4n+4) - \frac{1}{8}(n^2-8n+16)\right) + (-2)^n \\
&= 8c_0(-2)^n\left(\frac{1}{8}n^2 - 1\right) + (-2)^n \\
&= (-2)^n\left(c_0 n^2 + (-8c_0+1)\right) = c_0 n^2(-2)^n + (-8c_0+1)(-2)^n
\end{aligned}$$

So

$$-8c_0 + 1 = 0, \qquad \text{and so} \qquad c_0 = 1/8.$$

Thus $\hat{a}_n = \frac{1}{8}n^2(-2)^n$.

Therefore

$$a_n = h_n + \hat{a}_n = (\alpha_0 + \alpha_1 n)2^n + (\beta_0 + \beta_1 n)(-2)^n + \frac{1}{8}n^2(-2)^n.$$

PRO TIP. Check your particular solution by plugging in $a_n - (8a_{n-2} - 16a_{n-4} + (-2)^n)$ into wolframalpha.com. In the case of this problem, plug in

$$(1/8)n\hat{\ }2(-2)\hat{\ }n - (8(1/8)(n-2)\hat{\ }2(-2)\hat{\ }(n-2) - 16(1/8)(n-4)\hat{\ }2(-2)\hat{\ }(n-4) + (-2)\hat{\ }n)$$

You should get 0. (WHY??)

. . . .

(d) Find the general solution to

$$a_n = 8a_{n-2} - 16a_{n-4} + n^3.$$

Answer. Guess $\hat{a}_n = c_0 + c_1 n + c_2 n^2 + c_3 n^3$. Plugging in, we get

$$\begin{aligned} c_0 + c_1 n + c_2 n^2 + c_3 n^3 = \hat{a}_n &= 8a_{n-2} - 16a_{n-4} + n^3 \\ &= 8(c_0 + c_1(n-2) + c_2(n-2)^2 + c_3(n-2)^3) \\ &\qquad - 16(c_0 + c_1(n-4) + c_2(n-4)^2 + c_3(n-4)^3) + n^3. \end{aligned}$$

Plugging in 4 points (to get 4 equations for our 4 unknowns:)

$$\begin{aligned} n = 0:&\quad c_0 = 8(c_0 - 2c_1 + 4c_2 - 8c_3) - 16(c_0 - 4c_1 + 16c_2 - 64c_3), \\ &\text{so that } 0 = 9c_0 - 48c_1 + 224c_2 - 960c_3; \\ n = 1:&\quad c_0 + c_1 + c_2 + c_3 = 8(c_0 - c_1 + c_2 - c_3) - 16(c_0 - 3c_1 + 9c_2 - 27c_3) + 1 \\ &\text{so that } 1 = 9c_0 - 39c_1 + 137c_2 - 423c_3 - 1; \\ n = 2:&\quad c_0 + 2c_1 + 4c_2 + 8c_3 = 8c_0 - 16(c_0 - 2c_1 + 4c_2 - 8c_3) + 8 \\ &\text{so that } 8 = 9c_0 - 30c_1 + 68c_2 - 120c_3; \\ n = 3:&\quad c_0 + 3c_1 + 9c_2 + 27c_3 = 8(c_0 + c_1 + c_2 + c_3) - 16(c_0 - c_1 + c_2 - c_3) + 27 \\ &\text{so that } 27 = 9c_0 - 21c_1 + 17c_2 + 3c_3. \end{aligned}$$

Solving this system of linear equations gives

$$c_0 = 800/27, c_1 = 361/27, c_2 = 43/18, \quad \text{and} \quad c_3 = 1/6.$$

Thus $\hat{a}_n = \frac{800}{27} + \frac{361}{27}n + \frac{43}{18}n^2 + \frac{1}{6}n^3$, so that

$$a_n = h_n + \hat{a}_n = (\alpha_0 + \alpha_1 n)2^n + (\beta_0 + \beta_1 n)(-2)^n + \frac{800}{27} + \frac{361}{27}n + \frac{43}{18}n^2 + \frac{1}{6}n^3.$$

. . . .

(e) Pick an example of appropriate initial conditions for the sequence in part (c), and solve for the corresponding specific solution. Check your answer by computing the first 6 terms of the sequence both recursively and using your closed formula.

Answer. Let $a_0 = 3, a_1 = -17/4, a_2 = 22, a_3 = -9$. Then since

$$a_n = (\alpha_0 + \alpha_1 n)2^n + (\beta_0 + \beta_1 n)(-2)^n + \frac{1}{8}n^2(-2)^n,$$

plugging in $n = 0, 1, 2,$ and 3, respectively, gives

$$\begin{aligned}
3 &= a_0 = \alpha_0 + \beta_0 \\
-17/4 &= a_1 = (\alpha_0 + \alpha_1)2 + (\beta_0 + \beta_1)(-2) + \frac{1}{8}(-2) \\
&= 2\alpha_0 + 2\alpha_1 - 2\beta_0 - 2\beta_1 - 1/4 \\
22 &= a_2 = 4(\alpha_0 + 2\alpha_1) + 4(\beta_0 + 2\beta_1) + \frac{1}{8}(4)(4) \\
&= 4\alpha_0 + 8\alpha_1 + 4\beta_0 + 8\beta_1 + 2 \\
-9 &= a_3 = 8(\alpha_0 + 3\alpha_1) - 8(\beta_0 + 3\beta_1) + \frac{1}{8}(9)(-8) \\
&= 8\alpha_0 + 24\alpha_1 - 8\beta_0 - 24\beta_1 - 9;
\end{aligned}$$

so that

$$\begin{aligned}
3 &= \alpha_0 + \beta_0, \\
-4 &= 2\alpha_0 + 2\alpha_1 - 2\beta_0 - 2\beta_1, \\
20 &= 4\alpha_0 + 8\alpha_1 + 4\beta_0 + 8\beta_1, \text{ and} \\
0 &= 8\alpha_0 + 24\alpha_1 - 8\beta_0 - 24\beta_1.
\end{aligned}$$

Solving this system of linear equations gives

$$\alpha_0 = 0, \quad \alpha_1 = 1, \quad \beta_0 = 3, \quad \text{and} \quad \beta_1 = 0.$$

Thus

$$a_n = n2^n + 3(-2)^n + \frac{1}{8}n^2(-2)^n.$$

. . . .

Exercise 36.

(a) Explicitly compute the series for

$$\frac{1}{(1-x)^3} \quad \text{and} \quad \frac{1}{(1-x)^4}$$

by taking derivatives and rescaling appropriately. Conjecture what the general formula for the series for $1/(1-x)^n$.

Answer. We have

$$\begin{aligned}
\frac{1}{(1-x)^2} &= \frac{d}{dx}\frac{1}{1-x} = \sum_{k=0}^{\infty}\frac{d}{dx}x^k = \sum_{n=0}^{\infty}(n+1)x^n;\\
\frac{1}{(1-x)^3} &= \frac{1}{2}\frac{d}{dx}\frac{1}{(1-x)^2} = \sum_{k=0}^{\infty}(k+1)\frac{d}{dx}x^k\\
&= \sum_{n=0}^{\infty}\frac{(n+2)(n+1)}{2}x^n = \sum_{n=0}^{\infty}\binom{n+2}{2}x^n;\\
\frac{1}{(1-x)^4} &= \frac{1}{3}\frac{d}{dx}\frac{1}{(1-x)^3} = \sum_{k=0}^{\infty}\frac{(k+2)(k+1)}{2}\frac{d}{dx}x^k\\
&= \sum_{n=0}^{\infty}\frac{(n+3)(n+2)(n+1)}{3*2}x^n = \sum_{n=0}^{\infty}\binom{n+3}{3}x^n; \qquad \text{and}\\
\frac{1}{(1-x)^k} &= \sum_{n=0}^{\infty}\binom{n+k-1}{k-1}x^n.
\end{aligned}$$

. . . .

(b) Substitute $y = 3x$ into the series for $\frac{1}{1-y}$ to get the series for $\dfrac{1}{1-3x}$. What is the series for $1/(1-4x)$? What is the series for $1/(1+x)$?

Answer. We have

$$\begin{aligned}
\frac{1}{1-3x} &= \frac{1}{1-y}\bigg|_{y=3x} = \sum_{k=0}^{\infty}(3x)^k = \sum_{k=0}^{\infty}3^k x^k;\\
\frac{1}{1-4x} &= \frac{1}{1-y}\bigg|_{y=4x} = \sum_{k=0}^{\infty}(4x)^k = \sum_{k=0}^{\infty}4^k x^k;\\
\frac{1}{1+x} &= \frac{1}{1-y}\bigg|_{y=-x} = \sum_{k=0}^{\infty}(-x)^k = \sum_{k=0}^{\infty}(-1)^k x^k.
\end{aligned}$$

. . . .

(c) Substitute $y = x^3$ into the series for $\dfrac{1}{1-y}$ to get the series for $\dfrac{1}{1-x^3}$. What is the series for $1/(1-x^4)$? What is the series for $1/(1-2x^3)$?

Answer. We have

$$\begin{aligned}
\frac{1}{1-x^3} &= \frac{1}{1-y}\bigg|_{y=x^3} = \sum_{k=0}^{\infty}(x^3)^k = \sum_{k=0}^{\infty}x^{3k};\\
\frac{1}{1-x^4} &= \frac{1}{1-y}\bigg|_{y=x^4} = \sum_{k=0}^{\infty}(x^4)^k = \sum_{k=0}^{\infty}x^{4k};\\
\frac{1}{1-2x^3} &= \frac{1}{1-2y}\bigg|_{y=x^3} = \sum_{k=0}^{\infty}2^k y^k\bigg|_{y=x^3} = \sum_{k=0}^{\infty}2^k(x^3)^k = \sum_{k=0}^{\infty}2^k x^{3k}.
\end{aligned}$$

. . . .

(d) Use the fact that $\frac{d}{dx}(\ln(1-x)) = -\frac{1}{1-x}$ to compute the series for $\ln(1-x)$. (Integrate, and pick your "$+C$" to make it so that evaluating your series at $x = 0$ matches correctly with evaluating $\ln(1-x)$.)

$$\ln(1-x) = \int -\frac{1}{1-x}dx = \sum_{k=0}^{\infty} \int x^k dx = C + \sum_{k=0}^{\infty} \frac{-1}{k+1}x^{k+1} = C - x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + \cdots .$$

To solve for C, plug in $x = 0$: the left-hand side is

$$\ln(1-0) = \ln(1) = 0;$$

and the right-hand side is

$$C - 0 - \frac{0^2}{2} - \frac{0^3}{3} - \frac{0^4}{4} + \cdots = C.$$

Thus $C = 0$, so that

$$\ln(1-x) = \sum_{n=1}^{\infty} -\frac{1}{n}x^n.$$