Exercise 26.
(a) Consider strings of length 10 consisting of 1’s, 2’s, and/or 3’s.
   (i) How many of these are there (with no additional restrictions)?
      \textit{Answer:} \(3^{10}\) (three choices for each digit)
   (ii) How many of these are there that contain exactly three 1’s, two 2’s, and five 3’s?
      \textit{Answer.} We’re counting the anagrams of 1112233333:
      \[
      \frac{10!}{3!2!5!} = \binom{10}{3} \binom{10-3}{2} \binom{10-3-5}{5}.
      \]
(b) How many anagrams are there of MISSISSIPPI?
   \textit{Answer.} There are 4 S’s, 4 I’s, 2 P’s, and 1 M, therefore, there are
   \[
   \frac{11!}{4!4!2!1!} = \binom{11}{4} \binom{11-4}{4} \binom{11-4-4}{2} \binom{11-4-4-2}{1}
   \]
anagrams.
(c) Suppose you’ve got eight varieties of doughnuts to choose from at a doughnuts shop.
   (i) How many ways can you select 6 doughnuts?
      \textit{Answer.} We’re putting 6 indistinguishable objects (our choices) into 8 distinguishable boxes (the varieties of doughnuts):
      \[
      \binom{6 + (8-1)}{6}.
      \]
      (This is a “stars and bars” problem.)
   (ii) How many ways can you select a dozen (12) doughnuts?
      \textit{Answer.} We’re putting 12 indistinguishable objects (our choices) into 8 distinguishable boxes (the varieties of doughnuts):
      \[
      \binom{12 + (8-1)}{12}.
      \]
      (This is a “stars and bars” problem.)
   (iii) How many ways can you select a dozen doughnuts with at least one of each kind? [Hint: if there’s at least one of each kind, then how many choices are you really making?]
      \textit{Answer.} Since 8 choices have already been made, we’re left with putting 4 indistinguishable objects (our choices) into 8 distinguishable boxes (the varieties of doughnuts):
      \[
      \binom{4 + 8 - 1}{4}.
      \]
      (This is still a “stars and bars” problem, but accounting for choices already determined.)
(d) How many different combinations of pennies, nickels, dimes, quarters, and half dollars can a jar contain if it has 20 coins in it?

Answer. We’re putting 200 indistinguishable objects (instances of coins) into 5 distinguishable boxes (the varieties of coins):

\[
\binom{20 + 5 - 1}{20}.
\]

(This is a “stars and bars” problem.)

(e) Counting solutions.

(i) How many solutions are there to the equation \(x_1 + x_2 + x_3 = 10\), where \(x_1, x_2,\) and \(x_3\) are nonnegative integers?

Answer. We’re putting 10 indistinguishable objects (one unit at a time) into 3 distinguishable boxes (the value of the variables):

\[
\binom{10 + 3 - 1}{10}.
\]

(This is a “stars and bars” problem.)

(ii) How many solutions are there to the equation \(x_1 + x_2 + x_3 = 10\), where \(x_1, x_2,\) and \(x_3\) are strictly positive integers?

[Hint: See problem (c)(iii)]

Answer. Since 3 “units” have already been assigned (one to \(x_1\), one to \(x_2\), and one to \(x_3\)), we’re left with putting 7 indistinguishable objects (our units) into 3 distinguishable boxes (the variables):

\[
\binom{7 + 3 - 1}{7}.
\]

(This is still a “stars and bars” problem, but accounting for choices already determined.)

(iii) How many solutions are there to the equation \(x_1 + x_2 + x_3 \leq 10\), where \(x_1, x_2,\) and \(x_3\) are nonnegative integers? [Hint: Use an extra variable \(x_4\) such that \(x_1 + x_2 + x_3 + x_4 = 10\)]

Answer. The nonnegative integer solutions to \(x_1 + x_2 + x_3 \leq 10\) is the same as the nonnegative integer solutions to \(x_1 + x_2 + x_3 + x_4 = 10\) (where \(x_4 = 10 - (x_1 + x_2 + x_3)\)). So, similarly to the previous part, there are

\[
\binom{10 + 4 - 1}{10}
\]

solutions.

(This is still a “stars and bars” problem.)
Exercise 27.
(a) List the partitions of 6, both as box diagrams and as sequences.

Answer.

(b) How many ways are there to distribute 6 identical cookies into 6 identical lunch boxes, possibly leaving some empty?

Answer. This is the number of partitions of 6 with at most 6 parts, of which there are 12 (see part (a)).

(c) How many ways are there to distribute 6 identical snack bars into 4 identical lunch boxes, possibly leaving some empty?

Answer. This is the number of partitions of 6 with at most 4 parts, of which there are 9 (see part (a)).

(d) How many ways are there to distribute 4 identical apples into 6 identical lunch boxes, possibly leaving some empty?

Answer. This is the number of partitions of 4 with at most 6 parts, which is the same as the number of partitions with at most 4 parts, of which there are 5:
Exercise 28.
(a) Basic counting:

(i) How many ways are there to distribute 5 distinguishable objects into 3 distinguishable boxes, possibly leaving some empty?

\[ \text{Answer: } 3^5 \]

(ii) How many ways are there to distribute 5 indistinguishable objects into 3 distinguishable boxes, possibly leaving some empty?

\[ \binom{5+3-1}{3} \]

(iii) How many ways are there to distribute 5 distinguishable objects into 3 indistinguishable boxes, possibly leaving some empty?

\[ S(5, 3) + S(5, 2) + S(5, 1) \]

Answer. There are

\[ S(n, j) = \frac{1}{j!} \sum_{\ell=0}^{j-1} (-1)^\ell \binom{j}{\ell} (j - \ell)^n; \]

or you can count directly by cases as follows.

\[ S(5, 1) = 1: \text{ There is one way to put all 5 things into one box.} \]

\[ S(5, 2) = 5 + \binom{5}{2} = 15: \text{ If we split 5 things into two boxes then that split either looks like } \{a, b, c, d\}, \{e\} \text{ or } \{a, b, c\}, \{d, e\}. \]

In the first case, there are 5 ways to do this (5 ways to choose e); in the second case, there are \( \binom{5}{2} = 10 \) ways to choose \( d \) and \( e \).

\[ S(5, 3) = \binom{5}{2} + \frac{1}{2} \binom{5}{2} \binom{3}{2} = 25: \text{ If we split 5 things into three boxes then that split either looks like } \{a, b, c\}, \{d\}, \{e\} \text{ or } \{a, b\}, \{c, d\}, \{e\}. \]

In the first case, there are \( \binom{5}{2} \) to choose \( d \) and \( e \) (the order doesn’t matter since the boxes are indistinguishable—all we care about is that \( d \) and \( e \) get their own box, and the rest have to share a box). In the second case, there are \( \frac{1}{2} \binom{5}{2} \binom{3}{2} \) ways—pick \( a \) and \( b \), then pick \( c \) and \( d \), and then divide by the permutations of the first two sets (again since I can’t tell the difference between the boxes).

So

\[ S(5, 3) + S(5, 2) + S(5, 1) = 25 + 15 + 1. \]

(iv) How many ways are there to distribute 5 indistinguishable objects into 3 indistinguishable boxes, possibly leaving some empty?

Answer. The ways to do this are in bijection with integer partitions of 5 with at most 3 parts, so there are

\[ p_3(5) = \left| \{\{\cdot\cdot\cdot, \cdot\cdot, \cdot\cdot\}, \{\cdot\cdot, \cdot\cdot, \cdot\cdot\}, \{\cdot\cdot, \cdot\cdot, \cdot\cdot\}\} \right| = 5 \]

ways.
(v) How many ways are there to distribute 6 distinguishable objects into 4 indistinguishable boxes, possibly leaving some empty?

Answer:

\[ S(6, 4) + S(6, 3) + S(6, 2) + S(6, 1) \]

(vi) How many ways are there to distribute 6 distinguishable objects into 4 indistinguishable boxes so that each of the boxes contains at least one object? Answer: \( S(6, 4) \)

(b) How many ways are there to pack 8 identical DVDs into 5 indistinguishable boxes? How many ways to do this task so that each box contains at least one DVD?

Answer: In general,

\[ S(8, 5) + S(8, 4) + S(8, 3) + S(8, 2) + S(8, 1); \]

but only \( S(8, 5) \) if each box contains at least one DVD.

(c) How many ways are there to distribute 5 balls into 7 boxes if

(i) both the balls and boxes are labeled? \( \text{Answer: } 7^5 \)

(ii) the balls are labeled, but the boxes are unlabeled?

Answer. There are

\[ S(5, 7) + S(5, 6) + S(5, 5) + S(5, 4) + S(5, 3) + S(5, 2) + S(5, 1) = 0 + 0 + 1 + S(5, 4) + S(5, 3) + S(5, 2) + S(5, 1) \]

ways. Note that \( S(5, 7) = S(5, 6) = 0 \) since there is no way to leave no box empty when there are more boxes than balls.

(iii) the balls are unlabeled, but the boxes are labeled? \( \text{Answer: } \binom{5+7-1}{5} \)

(iv) both the balls and boxes are unlabeled?

Answer.

\[ p_7(5) = \left\{ \begin{array}{c}
\text{boxes}
\end{array} \right\} \]

(d) Repeat parts (i)–(iv) of part (c), adding the condition that each bucket can have at most one ball in it.

(i) both the balls and boxes are labeled:

Answer. There are \( 7 \times 6 \times 5 \times 4 \times 3 \) ways (pick 5 boxes from 7 in order without replacement).
(ii) the balls are labeled, but the boxes are unlabeled:

\textit{Answer}. Each ball has to go into a separate box, but we can’t tell the difference between the buckets. So there’s one way.

\hspace{1cm} ..........................................................................................................................

(iii) the balls are unlabeled, but the boxes are labeled:

\textit{Answer}. Each ball has to go into a separate box, and we can’t tell the difference between the balls, but we can tell which buckets have been filled. So there are \( \binom{5}{2} \) ways.

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(iv) both the balls and boxes are unlabeled:

\textit{Answer}. Each ball has to go into a separate box, and we can’t tell the difference between the balls or the buckets. So there’s one way.

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\textbf{Exercise 29.} Draw a tree-diagram that tells you how many ways to form the following results, and count the possible outcomes.

(a) Strings of 1’s and 0’s of length-four with three consecutive 0’s.

\textit{Answer}.

\hspace{1cm} ..........................................................................................................................
(b) Subsets of the set \(\{3, 7, 9, 11, 24\}\) whose elements sum to less than 28.

Answer.

Exercise 30.

(a) Permutations.

(i) Find a recurrence relation and initial conditions for the number of permutations of a set with \(n\) elements.

Answer. For each permutation of \(n - 1\), insert an \(n\). There are \(n\) ways to do this (insert at the beginning, after the first term, after the second term, \ldots, after the \(n - 1\) term):

\[ a_n = na_{n-1}. \]

This needs one initial condition. There is one permutation of one thing, so

\[ a_1 = 1. \]

(ii) Check your recurrence relation by iteratively calculating the first 5 terms of your sequence, and using the known closed formula for counting permutations.

Answer. We know that there are \(n!\) permutations of \(n\) elements.

\[
\begin{align*}
a_2 &= 2a_1 = 2 \times 1 = 2! \quad \checkmark \\
a_3 &= 3a_2 = 3 \times 2 \times 1 = 3! \quad \checkmark \\
a_4 &= 4a_3 = 4 \times 3 \times 2 \times 1 = 4! \quad \checkmark \\
a_5 &= 5a_4 = 5 \times 4 \times 3 \times 2 \times 1 = 5! \quad \checkmark 
\end{align*}
\]
(b) Bit strings.

(i) Find a recurrence relation and initial conditions for the number of bit strings of length \( n \) that contain a pair of consecutive 0s.

**Answer.** For every good bit string (a bit string containing at least one pair of consecutive 0's) or length \( n \), removing the last bit leave either a good or a bad bit string of length \( n - 1 \). For those that leave a good bit string, either the last digit is a 1 or a 0, so there are

\[
a_{n-1} \times 2
\]

of these.

For those that leave a bad bit string, this means that the \( n - 1 \) bit has to be a 0 and the \( n - 2 \) bit has to be a 1. The rest of the bits are free. Thus there are

\[
(\text{total } n - 3 \text{ strings}) - (\text{number of good } n - 3 \text{ strings}) = 2^{n-3} - a_{n-3}
\]

of these.

So

\[
a_n = 2a_{n-1} + 2^{n-3} - a_{n-3}.
\]

This requires three initial conditions. There are no good bit strings of length 1; there is 1 good string of length 2 (00), and there are three good strings of length 3 (000, 100, and 001). So

\[
a_1 = 0, \quad a_2 = 1, \quad a_3 = 3.
\]

(ii) Check your answer for \( n = 4 \) by iteratively using your recurrence relation, and then by listing the possibilities.

**Answer.** Decision tree:

![Decision Tree](image)

This shows that there are 8 good 4-strings. Alternatively,

\[
a_4 = 2a_3 + 2^1 - a_1 = 2 \times 3 + 2 - 0 = 8 \quad \checkmark
\]
(c) Climbing stairs.

(i) Find a recurrence relation and initial conditions for the number of ways to climb \( n \) stairs if the person climbing the stairs can take one stair or two stairs at a time.

**Answer.** Consider the last step taken. This is either two stairs or one. If it’s two, there are \( a_{n-2} \) ways to lead up to this; if it’s one, there are \( a_{n-1} \) ways to lead up to this. So

\[
a_n = a_{n-1} + a_{n-2}.
\]

If there’s only one stair, there’s one way to climb it. If there are two stairs, the person can take them one at a time, or both at once. So

\[
a_1 = 1 \quad \text{and} \quad a_2 = 2.
\]

(ii) Check your answer for \( n = 4 \) by iteratively using your recurrence relation, and by counting the number of these sequences by hand using a decision tree.

**Answer.**

\[
\begin{align*}
2 & \quad 2 \\
1 & \quad 1 & \quad 1 \\
1 & \quad 1 & \quad 1 \\
\end{align*}
\]

This shows that there are 5 ways to climb 4 stairs. Alternatively,

\[
\begin{align*}
a_3 &= a_2 + a_1 = 2 + 1 = 3 \\
a_4 &= a_3 + a_2 = 3 + 1 = 5
\end{align*}
\]

(iii) Calculate the number of ways to climb 8 stairs in this way.

**Answer.** Continuing from before:

\[
\begin{align*}
a_5 &= a_4 + a_3 = 5 + 3 = 8 \\
a_6 &= a_5 = 8 + 5 = 13 \\
a_7 &= a_6 + a_5 = 13 + 8 = 21 \\
a_8 &= a_7 + a_6 = 21 + 13 = 34.
\end{align*}
\]
(d) Tiling boards.

(i) Find a recurrence relation and initial conditions for the number of ways to completely cover a $2 \times n$ checkerboard with $1 \times 2$ dominoes. For example, if $n = 3$, one solution is

$2 \times 3$ checkerboard: covered with 3 dominoes: shorthand for same solution:

![Image of checkerboard and dominoes]

[Hint: Consider separately the coverings where the position in the top right corner of the checkerboard is covered by a domino positioned horizontally and where it is covered by a domino positioned vertically.]

**Answer.** If the top right corner is covered by a vertical domino, then the remainder of the board is a tiling of a $2 \times (n - 1)$ board, of which there are $a_{n-1}$ ways. If the top right corner is covered by a horizontal domino, then the bottom right corner is tiled by a horizontal domino. The rest of the board is a $2 \times (n - 2)$ board, of which there are $a_{n-2}$ ways to do this. So

$$a_n = a_{n-1} + a_{n-2}.$$  

There’s one way to tile a $2 \times 1$ board, and two ways to tile a $2 \times 2$ board, so

$$a_1 = 1 \quad \text{and} \quad a_2 = 2.$$

(ii) Check your answer for $n = 4$ by iteratively using your recurrence relation, and by counting the number of these sequences by hand.

**Answer.** The $2 \times 4$ tilings:

![Image of 2x4 tilings]

Alternatively,

$$a_3 = 2 + 1 = 3, \quad \text{so} \quad a_4 = 3 + 2 = 5. \quad \checkmark$$

(iii) How many ways are there to completely cover a $2 \times 6$ checkerboard with $1 \times 2$ dominoes?

**Answer.** Continuing from above,

$$a_5 = 5 + 3 = 8, \quad \text{so} \quad a_6 = 8 + 5 = 13.$$
(e) Increasing sequences

(i) Find a recurrence relation for the number of strictly increasing sequences of positive integers that have 1 as their first term and \( n \) as their last term, where \( n \) is a positive integer. That is, sequences \( a_1, a_2, \ldots, a_k \), where \( a_1 = 1, a_k = n \), and \( a_j < a_{j+1} \) for \( j = 1, 2, \ldots, k - 1 \).

**Answer.** Let \( H_n \) be the number of these sequences. Each sequence ending in \( n \) either has \( n-1 \) in it or it doesn’t. By removing the last term from a sequence that has \( n-1 \) in it, you’re left with an increasing sequence starting at 1 and ending at \( n-1 \), of which there are \( H_{n-1} \). If the sequence doesn’t have an \( n-1 \) in it, then replacing \( n \) with \( n-1 \) leaves an increasing sequence starting at 1 and ending at \( n-1 \), of which there are \( H_{n-1} \). So

\[
H_n = 2H_{n-1}.
\]

There is one sequence starting at 1 and ending at 2, so \( H_2 = 1 \) (it doesn’t make sense to start with \( H_2 \)).

(ii) Check your answer for \( n = 4 \) by iteratively using your recurrence relation, and by counting the number of these sequences by hand using a decision tree.

**Answer.**

![Decision Tree](image)

This says there are four such sequences. Alternatively,

\[
a_3 = 2a_2 = 2, \quad \text{and so} \quad a_4 = 2a_3 = 2 \times 2 = 4 \quad \checkmark.
\]

(iii) Explain why there are infinitely many such sequences if we replace “strictly increasing” with “weakly increasing” in part (i), i.e. turn “<” into “\( \leq \)”.

**Answer.** This will include sequences like \( 1, 1, \ldots, 1, n \) arbitrarily many 1’s.