SOLUTIONS
Homework 6
Due 3/20/19

Exercise 26.

(a) Consider strings of length 10 consisting of 1's, 2's, and/or 3's.

(i) How many of these are there (with no additional restrictions)?

Answer: 3^{10} (three choices for each digit)

(ii) How many of these are there that contain exactly three 1's, two 2's, and five 3's?

Answer. We're counting the anagrams of 1112233333:

$$\frac{10!}{3!2!5!} = \binom{10}{3}\binom{10-3}{2}\binom{10-3-5}{5}.$$

. . . .

(b) How many anagrams are there of MISSISSIPPI?

Answer. There are 4 S's, 4 I's, 2 P's, and 1 M, therefore, there are

$$\frac{11!}{4!4!2!1!} = \binom{11}{4}\binom{11-4}{4}\binom{11-4-4}{2}\binom{11-4-4-2}{1}$$

anagrams.

. . . .

(c) Suppose you've got eight varieties of doughnuts to choose from at a doughnuts shop.

(i) How many ways can you select 6 doughnuts?

Answer. We're putting 6 indistinguishable objects (our choices) into 8 distinguishable boxes (the varieties of doughnuts):

$$\binom{6+(8-1)}{6}.$$

(This is a "stars and bars" problem.)

. . . .

(ii) How many ways can you select a dozen (12) doughnuts?

Answer. We're putting 12 indistinguishable objects (our choices) into 8 distinguishable boxes (the varieties of doughnuts):

$$\binom{12+(8-1)}{12}.$$

(This is a "stars and bars" problem.)

. . . .

(iii) How many ways can you select a dozen doughnuts with at least one of each kind?
[Hint: if there's at least one of each kind, then how many choices are you really making?]

Answer. Since 8 choices have already been made, we're left with putting 4 indistinguishable objects (our choices) into 8 distinguishable boxes (the varieties of doughnuts):

$$\binom{4+8-1}{4}.$$

(This is still a "stars and bars" problem, but accounting for choices already determined.)

. . . .

(d) How many different combinations of pennies, nickels, dimes, quarters, and half dollars can a jar contain if it has 20 coins in it?

Answer. We're putting 200 indistinguishable objects (instances of coins) into 5 distinguishable boxes (the varieties of coins):

$$\binom{20+5-1}{20}.$$

(This is a "stars and bars" problem.)

. . . .

(e) Counting solutions.

(i) How many solutions are there to the equation $x_1+x_2+x_3=10$, where x_1, x_2, and x_3 are nonnegative integers?

Answer. We're putting 10 indistinguishable objects (one unit at a time) into 3 distinguishable boxes (the value of the variables):

$$\binom{10+3-1}{10}.$$

(This is a "stars and bars" problem.)

. . . .

(ii) How many solutions are there to the equation $x_1+x_2+x_3=10$, where x_1, x_2, and x_3 are strictly positive integers?
[Hint: See problem (c)(iii)]

Answer. Since 3 "units" have already been assigned (one to x_1, one to x_2, and one to x_3), we're left with putting 7 indistinguishable objects (our units) into 3 distinguishable boxes (the variables):

$$\binom{7+3-1}{7}.$$

(This is still a "stars and bars" problem, but accounting for choices already determined.)

. . . .

(iii) How many solutions are there to the equation $x_1+x_2+x_3 \leq 10$, where x_1, x_2, and x_3 are nonnegative integers? [Hint: Use an extra variable x_4 such that $x_1+x_2+x_3+x_4=10$]

Answer. The nonnegative integer solutions to $x_1+x_2+x_3 \leq 10$ is the same as the nonnegative integer solutions to $x_1+x_2+x_3+x_4=10$ (where $x_4 = 10-(x_1+x_2+x_3)$). So, similarly to the previous part, there are

$$\binom{10+4-1}{10} \qquad \text{solutions.}$$

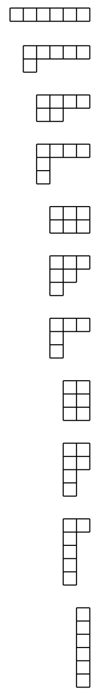
(This is still a "stars and bars" problem.)

. . . .

Exercise 27.

(a) List the partitions of 6, both as box diagrams and as sequences.

Answer.



(6)
$(5,1)$
$(4,2)$
$(4,1,1)$
$(3,3)$
$(3,2,1)$
$(3,1,1)$
$(2,2,2)$
$(2,2,1,1)$
$(2,1,1,1,1)$
$(1,1,1,1,1,1)$

. . . .

(b) How many ways are there to distribute 6 identical cookies into 6 identical lunch boxes, possibly leaving some empty?

Answer. This is the number of partitions of 6 with at most 6 parts, of which there are 12 (see part (a)).

. . . .

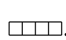
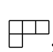
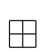
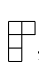
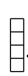
(c) How many ways are there to distribute 6 identical snack bars into 4 identical lunch boxes, possibly leaving some empty?

Answer. This is the number of partitions of 6 with at most 4 parts, of which there are 9 (see part (a)).

. . . .

(d) How many ways are there to distribute 4 identical apples into 6 identical lunch boxes, possibly leaving some empty?

Answer. This is the number of partitions of 4 with at most 6 parts, which is the same as the number of partitions with at most 4 parts, of which there are 5:

, ,  , and .

. . . .

Exercise 28.

(a) Basic counting:

(i) How many ways are there to distribute 5 distinguishable objects into 3 distinguishable boxes, possibly leaving some empty? *Answer:* 3^5

(ii) How many ways are there to distribute 5 indistinguishable objects into 3 distinguishable boxes, possibly leaving some empty? *Answer:* $\binom{5+3-1}{5}$

(iii) How many ways are there to distribute 5 distinguishable objects into 3 indistinguishable boxes, possibly leaving some empty?

Answer. There are

$$S(5,3) + S(5,2) + S(5,1)$$

ways, where $S(n,j)$ is the Stirling number of the second kind.
To compute this number explicitly, you can either use the formula

$$S(n,j) = \frac{1}{j!}\sum_{\ell=0}^{j-1}(-1)^\ell \binom{j}{\ell}(j-\ell)^n;$$

or you can count directly by cases as follows.

$S(5,1) = 1$: There is one way to put all 5 things into one box.

$S(5,2) = 5 + \binom{5}{2} = 15$: If we split 5 things into two boxes then that split either looks like

$$\{a,b,c,d\},\{e\} \qquad \text{or} \qquad \{a,b,c\},\{d,e\}.$$

In the first case, there are 5 ways to do this (5 ways to choose e); in the second case, there are $\binom{5}{2} = 10$ ways to choose d and e.

$S(5,3) = \binom{5}{2} + \frac{1}{2}\binom{5}{2}\binom{3}{2} = 25$: If we split 5 things into three boxes then that split either looks like

$$\{a,b,c\},\{d\},\{e\} \qquad \text{or} \qquad \{a,b\},\{c,d\},\{e\}.$$

In the first case, there are $\binom{5}{2}$ to choose d and e (the order doesn't matter since the boxes are indistinguishable–all we care about is that d and e get their own box, and the rest have to share a box). In the second case, there are $\frac{1}{2}\binom{5}{2}\binom{3}{2}$ ways–pick a and b, then pick c and d, and then divide by the permutations of the first two sets (again since I can't tell the difference between the boxes).

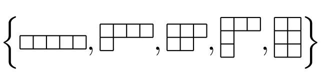
So

$$S(5,3) + S(5,2) + S(5,1) = 25 + 15 + 1.$$

. . . .

(iv) How many ways are there to distribute 5 indistinguishable objects into 3 indistinguishable boxes, possibly leaving some empty?

Answer. The ways to do this are in bijection with integer partitions of 5 with at most 3 parts, so there are

$p_3(5) = \Big|\Big\{$  $\Big\}\Big| = 5$

ways

. . . .

(v) How many ways are there to distribute 6 distinguishable objects into 4 indistinguishable boxes, possibly leaving some empty?

Answer.

$$S(6,4) + S(6,3) + S(6,2) + S(6,1)$$

. . . .

(vi) How many ways are there to distribute 6 distinguishable objects into 4 indistinguishable boxes so that each of the boxes contains at least one object? *Answer:* $S(6,4)$

(b) How many ways are there to pack 8 identical DVDs into 5 indistinguishable boxes? How many ways to do this task so that each box contains at least one DVD?

Answer. In general,

$$S(8,5) + S(8,4) + S(8,3) + S(8,2) + S(8,1);$$

but only $S(8,5)$ if each box contains at least one DVD.

. . . .

(c) How many ways are there to distribute 5 balls into 7 boxes if

(i) both the balls and boxes are labeled? *Answer:* 7^5

(ii) the balls are labeled, but the boxes are unlabeled?

Answer. There are

$$S(5,7)+S(5,6)+S(5,5)+S(5,4)+S(5,3)+S(5,2)+S(5,1) = 0+0+1+S(5,4)+S(5,3)+S(5,2)+S(5,1)$$

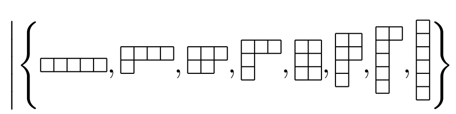
ways. Note that $S(5,7) = S(5,6) = 0$ since there is no way to leave no box empty when there are more boxes than balls.

. . . .

(iii) the balls are unlabeled, but the boxes are labeled? *Answer:* $\binom{5+7-1}{5}$

(iv) both the balls and boxes are unlabeled?

Answer.

$p_7(5) =$ 

. . . .

(d) Repeat parts (i)–(iv) of part (c), adding the condition that each bucket can have at *most* one ball in it.

(i) both the balls and boxes are labeled:

Answer. There are $7*6*5*4*3$ ways (pick 5 boxes from 7 in order without replacement).

. . . .

(ii) the balls are labeled, but the boxes are unlabeled:

Answer. Each ball has to go into a separate box, but we can't tell the difference between the buckets. So there's one way.

. . . .

(iii) the balls are unlabeled, but the boxes are labeled:

Answer. Each ball has to go into a separate box, and we can't tell the difference between the balls, but we can tell which buckets have been filled. So there are $\binom{7}{5}$ ways.

. . . .

(iv) both the balls and boxes are unlabeled:

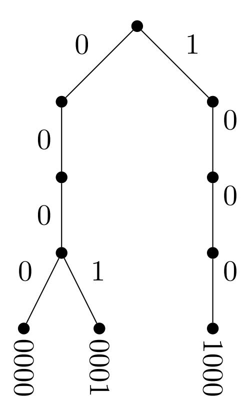
Answer. Each ball has to go into a separate box, and we can't tell the difference between the balls or the buckets. So there's one way.

. . . .

Exercise 29. Draw a tree-diagram that tells you how many ways to form the following results, and count the possible outcomes.

(a) Strings of 1's and 0's of length-four with three consecutive 0's.

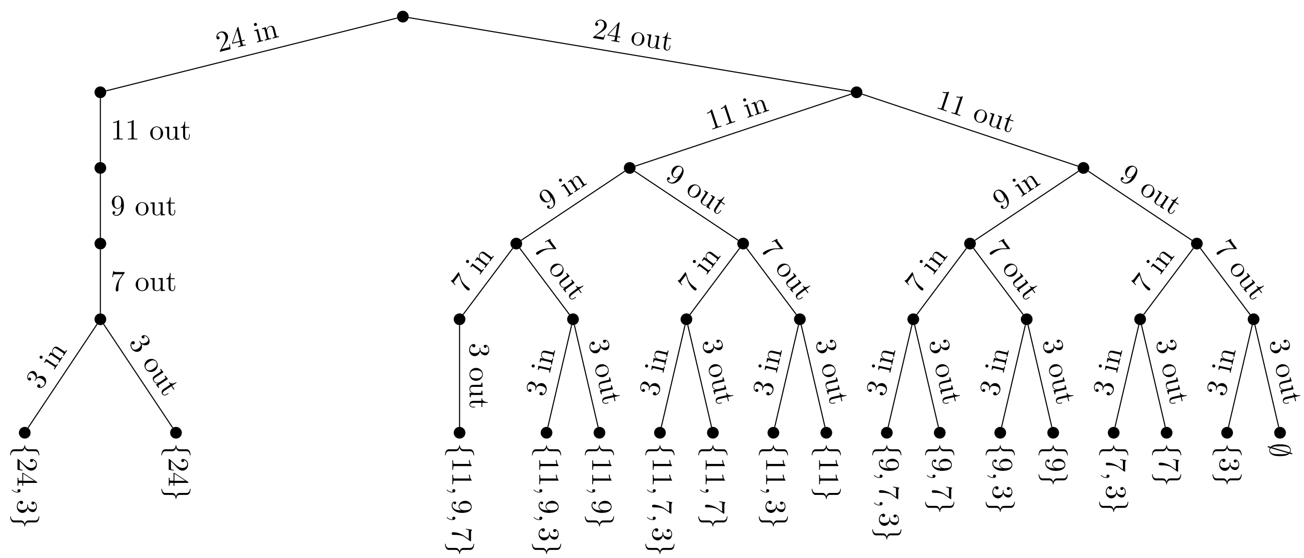
Answer.



. . . .

(b) Subsets of the set $\{3, 7, 9, 11, 24\}$ whose elements sum to less than 28.

Answer.



. . . .

To check your answers: (a) 3; (b) 17.

Exercise 30.

(a) Permutations.

(i) Find a recurrence relation and initial conditions for the number of permutations of a set with n elements.

Answer. For each permutation of $n-1$, insert an n. There are n ways to do this (insert at the beginning, after the first term, after the second term, . . . , after the $n-1$ term):

$$a_n = na_{n-1}.$$

This needs one initial condition. There is one permutation of one thing, so

$$a_1 = 1.$$

. . . .

(ii) Check your recurrence relation by iteratively calculating the first 5 terms of your sequence, and using the known closed formula for counting permutations.

Answer. We know that there are $n!$ permutations of n elements.

$$a_2 = 2a_1 = 2 * 1 = 2! \quad \checkmark$$
$$a_3 = 3a_2 = 3 * 2 * 1 = 3! \quad \checkmark$$
$$a_4 = 4a_3 = 4 * 3 * 2 * 1 = 4! \quad \checkmark$$
$$a_5 = 5a_4 = 5 * 4 * 3 * 2 * 1 = 5! \quad \checkmark$$

. . . .

(b) Bit strings.

(i) Find a recurrence relation and initial conditions for the number of bit strings of length n that contain a pair of consecutive 0s.

Answer. For every good bit string (a bit string containing at least one pair of consecutive 0's) or length n, removing the last bit leave either a good or a bad bit string of length $n-1$. For those that leave a good bit sting, either the last digit is a 1 or a 0, so there are

$$a_{n-1} * 2 \qquad \text{of these.}$$

For those that leave a bad bit string, this means that the $n-1$ bit has to be a 0 and the $n-2$ bit has to be a 1. The rest of the bits are free. Thus there are

$$(\text{total } n-3 \text{ strings}) - (\text{number of good } n-3 \text{ strings}) = 2^{n-3} - a_{n-3} \qquad \text{of these.}$$

So

$$a_n = 2a_{n-1} + 2^{n-3} - a_{n-3.}$$

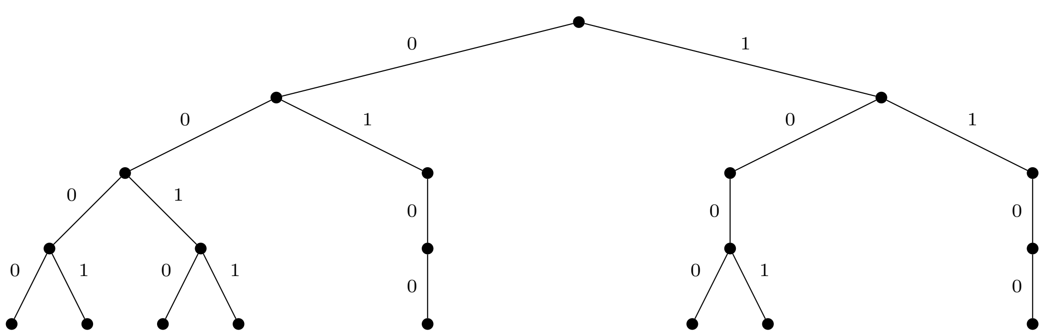
This requires three initial conditions. There are no good bit strings of length 1; there is 1 good string of length 2 (00), and there are three good strings of length 3 (000, 100, and 001). So

$$a_1 = 0, \quad a_2 = 1, \quad a_3 = 3.$$

. . . .

(ii) Check your answer for $n = 4$ by iteratively using your recurrence relation, and then by listing the possibilities.

Answer. Decision tree:



This shows that there are 8 good 4-strings. Alternatively,

$$a_4 = 2a_3 + 2^1 - a_1 = 2 * 3 + 2 - 0 = 8 \qquad \checkmark$$

. . . .

(c) Climbing stairs.

(i) Find a recurrence relation and initial conditions for the number of ways to climb n stairs if the person climbing the stairs can take one stair or two stairs at a time.

Answer. Consider the last step taken. This is either two stairs or one. If it's two, there are a_{n-2} ways to lead up to this; if it's one, there are a_{n-1} ways to lead up to this. So

$$a_n = a_{n-1} + a_{n-2}.$$

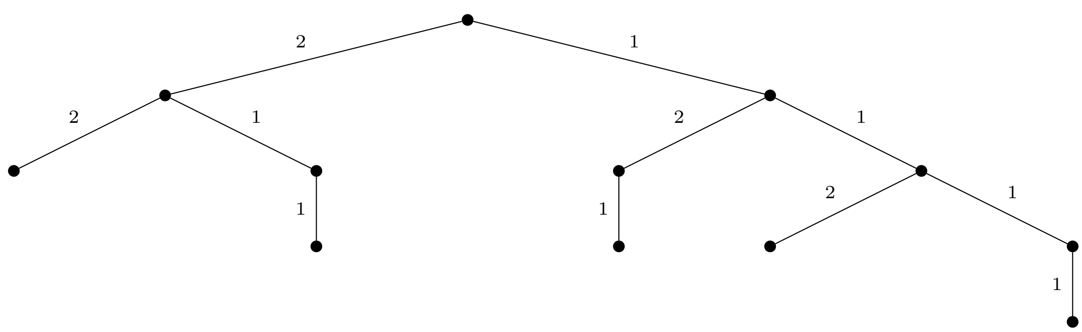
If there's only one stair, there's one way to climb it. If there are two stairs, the person can take them one at a time, or both at once. So

$$a_1 = 1 \qquad \text{and} \qquad a_2 = 2.$$

. . . .

(ii) Check your answer for $n = 4$ by iteratively using your recurrence relation, and by counting the number of these sequences by hand using a decision tree.

Answer.



This shows that there are 5 ways to climb 4 stairs. Alternatively,

$$a_3 = a_2 + a_1 = 2 + 1 = 3$$
$$a_4 = a_3 + a_2 = 3 + 1 = 5 \qquad \checkmark$$

. . . .

(iii) Calculate the number of ways to climb 8 stairs in this way.

Answer. Continuing from before:

$$a_5 = a_4 + a_3 = 5 + 3 = 8 \; a_6 \qquad = a_5 + a_4 = 8 + 5 = 13$$
$$a_7 = a_6 + a_5 = 13 + 8 = 21$$
$$a_8 = a_7 + a_6 = 21 + 13 = 34.$$

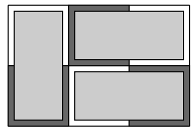
. . . .

(d) Tiling boards.

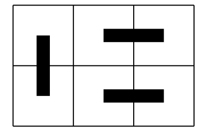
(i) Find a recurrence relation and initial conditions for the number of ways to completely cover a $2 \times n$ checkerboard with 1×2 dominoes. For example, if $n = 3$, one solution is

2×3 checkerboard:

covered with 3 dominoes:



shorthand for same solution:



[Hint: Consider separately the coverings where the position in the top right corner of the checkerboard is covered by a domino positioned horizontally and where it is covered by a domino positioned vertically.]

Answer. If the top right corner is covered by a vertical domino, then the remainder of the board is a tiling of a $2 \times (n-1)$ board, of which there are a_{n-1} ways. If the top right corner is covered by a horizontal domino, then the bottom right corner is tiled by a horizontal domino. The rest of the board is a $2 \times (n-2)$ board, of which there are a_{n-2} ways to do this. So

$$a_n = a_{n-1} + a_{n-2}.$$

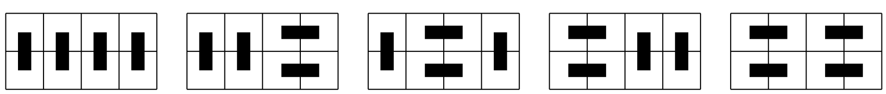
There's one way to tile a 2×1 board, and two ways to tile a 2×2 board, so

$$a_1 = 1 \qquad \text{and} \qquad a_2 = 2.$$

. . . .

(ii) Check your answer for $n = 4$ by iteratively using your recurrence relation, and by counting the number of these sequences by hand.

Answer. The 2×4 tilings:



Alternatively,

$$a_3 = 2 + 1 = 3, \quad \text{so} \quad a_4 = 3 + 2 = 5. \qquad \checkmark$$

. . . .

(iii) How many ways are there to completely cover a 2×6 checkerboard with 1×2 dominoes?

Answer. Continuing from above,

$$a_5 = 5 + 3 = 8, \quad \text{so} \quad a_6 = 8 + 5 = 13.$$

. . . .

(e) Increasing sequences

(i) Find a recurrence relation for the number of strictly increasing sequences of positive integers that have 1 as their first term and n as their last term, where n is a positive integer. That is, sequences $a_1, a_2, \ldots, a_k$, where $a_1 = 1$, $a_k = n$, and $a_j < a_{j+1}$ for $j = 1, 2, \ldots, k-1$.

Answer. Let H_n be the number of these sequences. Each sequence ending in n either has $n-1$ in it or it doesn't. By removing the last term from a sequence that has $n-1$ in it, you're left with an increasing sequence starting at 1 and ending at $n-1$, of which there are H_{n-1}. If the sequence doesn't have an $n-1$ in it, then replacing n with $n-1$ leaves an increasing sequence starting at 1 and ending at $n-1$, of which there are H_{n-1}. So

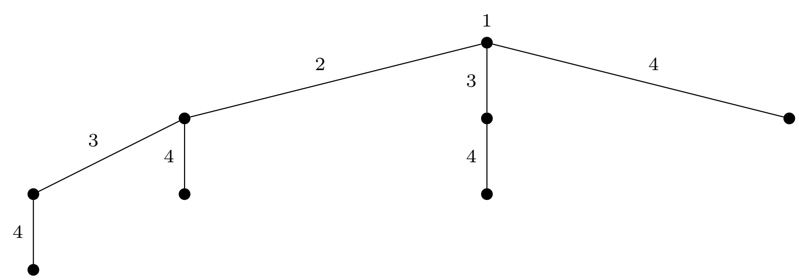
$$H_n = 2H_{n-1}.$$

There is one sequence starting at 1 and ending at 2, so $H_2 = 1$ (it doesn't make sense to start with H_2).

. . . .

(ii) Check your answer for $n = 4$ by iteratively using your recurrence relation, and by counting the number of these sequences by hand using a decision tree.

Answer.



This says there are four such sequences. Alternatively,

$$a_3 = 2a_2 = 2, \quad \text{and so} \quad a_4 = 2a_3 = 2 * 2 = 4 \quad \checkmark.$$

. . . .

(iii) Explain why there are infinitely many such sequences if we replace "strictly increasing" with "weakly increasing" in part (i), i.e. turn "$<$" into "$\leq$".

Answer. This will include sequences like $\underbrace{1, 1, \ldots, 1}_{\text{arbitrarily many 1's}}, n$

. . . .