SOLUTIONS
Homework 5
Due 3/6/19

Exercise 22.

(a) Consider the set $\{a, b, c\}$. For each of the following, (A) list the objects described, (B) give a formula that tells you how many you should have listed, and (C) verify that the formula and the list agree.

(i) Permutations of $\{a, b, c\}$.

Answer.
(A) $abc, acb, bac, bca, cab, cba$
(B) $P(3, 3) = 3!$
(C) $3! = 6$, and there are 6 items in (A).

. . . .

(ii) Two-permutations of $\{a, b, c\}$.

Answer.
(A) ab, ac, ba, bc, ca, cb
(B) $P(3, 2) = 3 * 2$
(C) $3 * 2 = 6$, and there are 6 items in (A).

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(iii) Size-two subsets of $\{a, b, c\}$.

Answer.
(A) $\{a, b\}, \{a, c\}, \{b, c\}$
(B) $\binom{3}{2} = 3 * 2/2!$
(C) $3 * 2/2! = 3$, and there are 3 items in (A).

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(b) For each of the following, classify the problem as a permutation or a combination problem or neither, and give an answer using an unsimplified formula. (Answers should look, for example, like $5 * 4$ or $5!/2!$ instead of $P(5, 4)$ or 20.)

(i) In how many different orders can five runners finish a race if no ties are allowed?

Answer. This is a "permutation" problem, and there are $\boxed{5! = 5 * 4 * 3 * 2 * 1}$ ways.

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(ii) How many strings of 1's and 0's of length seven have exactly three 1's?

Answer. This is a "combination" problem: Choosing the places where the 1's go gives $\binom{7}{3} = \boxed{\frac{7!}{4!3!}}$ strings.

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(iii) How many strings of 1's and 0's of length seven have three or fewer 1's?

Answer. This is a "combination" problem, to be computed in cases depending on *exactly* how many 1's the sequence has:

$$\binom{7}{3} + \binom{7}{2} + \binom{7}{1} + \binom{7}{0} = \boxed{7!\left(\frac{1}{4!3!} + \frac{1}{5!2!} + \frac{1}{6!1!} + \frac{1}{7!0!}\right)}.$$

. . . .

(iv) How many three-digit numbers are there with no 1's? (a three-digit number is something like 144 or 009 or 053)

Answer. This is neither combination nor permutation. This is just product rule: there are nine choices for each digit, giving $\boxed{9^3}$ in total.

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(v) How many three-digit numbers are there with no digits repeated?

Answer. This is a "permutation" problem, and there are $P(10,3) = \boxed{10 * 9 * 8}$ of these.

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(c) For each of the following, provide your answers in an unsimplified form, and justify.

(i) A six-sided dice is rolled 5 times. How many ways could it turn out that a value greater than 4 (i.e. 5 or 6) is rolled exactly twice? *(Hint: first pick which rolls are from* $\{5,6\}$ *(this implies which rolls are from* $\{1,2,3,4\}$ *for free), and then pick the values for the* $\{5,6\}$*-valued rolls, and finally pick the values for the other rolls (the* $\{1,2,3,4\}$*-valued rolls).)*

Answer. First choose when the high rolls happen: $\binom{5}{2} = \frac{5*4}{2}$ possibilities.
Then choose how the high rolls go: $2 * 2$ possibilities.
Then choose how the low rolls go: $4 * 4 * 4 * 4$ possibilities.
(For example, a roll sequence like $5, 2, 2, 6, 1$ is in the category of
roll sequence goes HLLHL;
high rolls go 5 then 6;
low rolls go 2 then 2 then 1.)
Now use product rule to combine: $\frac{5*4}{2}(2 * 2)(4 * 4 * 4 * 4)$.

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(ii) If 10 men and 10 women show up for one team of an intramural basketball game, how many ways can you pick 5 people to play for one team if there must be at least one person of each gender on the team?

Answer. Break this into the cases of how the gender balance works: 4 men and 1 women, 3 men and 2 women, 2 men and 3 women, or 1 man and 4 women. Then use product rule in each of those cases to get

$$\binom{10}{4}\binom{10}{1} + \binom{10}{3}\binom{10}{2} + \binom{10}{2}\binom{10}{3} + \binom{10}{1}\binom{10}{4}$$
$$= \left(\frac{10!}{4!1!}\right)^2 + \left(\frac{10!}{3!2!}\right)^2 + \left(\frac{10!}{3!2!}\right)^2 + \left(\frac{10!}{4!1!}\right)^2$$

. . . .

(iii) How many ways are there for 5 women and 2 men to stand in line? Now how many ways are there for them to stand in line if the two men don't stand next to each other? (The men and the women are distinct individuals.)

Answer.
Ways are there for 5 women and 2 men to stand in line:
This is just 7 people standing in line, which has 7! possibilities.

If the two men don't stand next to each other:
Pick where then men are standing, of which there are $\binom{7}{2} - 6$ possibilities.
Then pick the order of the women – 5! – and the order of the men – 2.
Combine using product rule: $(\binom{7}{2} - 6)5! * 2$.

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(d) For each of the following identities, (A) explain in words why it makes sense given what it represents, and then (B) verify it algebraically using the formulas for permutation or combination. *(For example, an answer for (A) might start out looking like "$P(n, 1)$ means. . . ", and an answer for part (B) should look like a calculation that starts with "$P(n, 1) = \dots$ ".)*

(i) $P(n, 1) = n$

Answer.

(A) This is the number of ways to pick one thing out of n, of which there are n possibilities. Order doesn't come in to play, since there's only one thing.

(B) $P(n, 1) = \frac{n!}{(n-1)!} = n$

. . . .

(ii) $P(n, 0) = 1$

Answer.

(A) This is the number of ways to pick nothing out of n, of which there is only one possibility.

(B) $P(n, 0) = \frac{n!}{n!} = 1$

. . . .

(iii) $P(n, k + 1) = P(n, k) * (n - k)$

Answer.

(A) $P(n, k + 1)$ is the number of ways to choose $k + 1$ things from n in order. Since it's in order, you can choose the first k things first, and then choose the last thing separately. After k things, there are $n - k$ to choose from.

(B) $P(n, k + 1) = \frac{n!}{(n-(k+1))!} = \frac{n!}{(n-k)!/(n-k)} P(n, k) * (n - k)$

. . . .

(iv) $\binom{n}{1} = n$

Answer.

(A) This is the number of ways to pick one thing out of n, of which there are n possibilities.

(B) $\binom{n}{1} = \frac{n!}{(n-1)!1!} = n$

. . . .

(v) $\binom{n}{n} = 1$

Answer.

(A) This is the number of ways to choose everything all at once out of a group of n. There is one way to do this – pick everything.

(B) $\binom{n}{n} = \frac{n!}{(n-n)!n!} = 1$ (since $0! = 1$)

. . . .

(vi) $\binom{n}{0} = 1$

Answer.
(A) This is the number of ways to choose nothing from n things. There is one way to do that - don't take anything.
(B) $\binom{n}{0} = \frac{n!}{(n-0)!0!} = 1$ (again, since $0! = 1$)

. . . .

(vii) $\binom{n}{k} = \binom{n}{n-k}$

Answer.
(A) This is because choosing k things from n is the same as picking $n-k$ things from n to exclude.
(B) $\binom{n}{k} = \frac{n!}{(n-k)!k!} = \frac{n!}{k!(n-k)!} = \binom{n}{n-k}$

. . . .

Exercise 23. For each of the following, be sure to include how the pigeonhole principle or its generalized version apply in your justifications, or why neither of them do.

(1) The lights have gone out and you're digging through an unorganized sock drawer filled with unmatched black socks and brown socks (otherwise roughly identical).
   (a) If you're pulling them out at random, how many socks do you need to take out to ensure you have a matching pair if there are 10 of each kind of sock? How about if there are 20 of each? 100 of each?

   Answer. It doesn't matter how many socks you have to choose from as long as you have enough to apply generalized pigeonhole principle. In any case, the colors are the boxes and the socks you've picked are the items, and so you want to solve $\lceil n/2 \rceil \geq 2$ for the minimal n: $n = 3$.

   . . . .

   (b) Again pulling at random, how many socks do you need to take out to ensure you have a matching brown pair if there are 10 of each kind of sock? How about if there are 20 of each? 100 of each?

   Answer. This is not pigeonhole principle since you need brown in particular. This is asking to guarantee that a *specific* box has at least 2 object. So you need to take into account that you might pull out all of the black socks before you get a single brown sock. Therefore you need to pick 12 if there are 10 of each kind of sock, 22 if there are 20 of each, and 102 if there are 100 of each.

   . . . .

(2) Explain why, out of any set of four integers, at least two have the same remainder when divided by 3.

   Answer. Here the elements are the integers and the remainders are the boxes. There are only three possible remainders when dividing by 3: 0, 1, and 2. Since you've picked 4 objects, the pigeonhole principle tells us at least 2 of them go in the same box, i.e. that 2 of them have the same remainder.

   . . . .

(3) A recent estimate showed that the US and Canada together (which share the country code +1) have approximately 134,000,000 phone lines in use. What is the minimum number of area codes needed to make that possible?

Answer. The boxes are the area codes and the 7-digit phone numbers are the objects. You have to ensure that there are enough of them so that no more than then number of possible 7-digit phone numbers are forced into the same area code. Since there are $8*10^6$ possible 7-digit phone numbers, you want to solve for the minimal k such that

$$\lceil (134*10^6)/k \rceil \leq 7*10^6.$$

This is $k = 20$.

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(4) Let $f : A \to B$ be a function between finite sets such that $|A| > |B|$. Explain why f cannot possibly be injective. (Consider the sizes of the preimages $\{f^{-1}(b) \mid b \in B\}$.)

Answer. Here, the elements of B are the boxes and the elements of A are the objects. A function f is an assignment of what objects go into what boxes (an object a is in box b if a is in the preimage of b, i.e. $f(a) = b$). Since there are more objects than boxes, pigeonhole principle tells us that some box has more than one object in it, i.e. some element of b has more than one element of a in its preimage. Therefore the function is not injective.

. . . .

(5) Explain why, in any sequence of n consecutive integers, at least one of them must be divisible by n. (Start with, say, $n = 4$ as an example.)

Answer. Like in part (c), the remainders are the boxes and the consecutive integers are the boxes.

Let $a, a+1, a+2, \ldots, a+n-1$ be a collection of consecutive ingeters. And suppose the least of these integers has remainder r when divided by n (so that $a = kn+r$, with $0 \leq r < n$). Now suppose for the sake of contradiction that none of them have a remainder of 0.

Note that when you increase an integer z by 1, you increase its remainder by 1; unless $z+1$ is divisible by n, in which case you drop it to 0. Since none of the consecutive integers is divisible by n, not only do none of them have a remainder of 0, but moving up the chain, you never drop the remainder to 0. So the integers have distinct remainders.

However, if none of the integers are divisible by n, then we're only using $n-1$ out of the n boxes. So pigeonhole principle tells us that some box has at least 2 elements, i.e. there are two of these integers that have the same remainder when divided by n. This is a contradiction.

Thus one of any n consecutive integers must be divisible by n.

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Exercise 24.

(a) Expand $(x+y)^5$ and $(x+y)^8$ using the binomial theorem.

Answer. By the binomial theorem, we have

$$\begin{aligned}
(x+y)^5 &= \binom{5}{0}x^5 + \binom{5}{1}x^4y + \binom{5}{2}x^3y^2 + \binom{5}{3}x^2y^3 + \binom{5}{4}xy^4 + \binom{5}{5}y^5 \\
&= x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5; \text{ and} \\
(x+y)^8 &= \binom{8}{0}x^8 + \binom{8}{1}x^7y + \binom{8}{2}x^6y^2 + \binom{8}{3}x^5y^3 + \binom{8}{4}x^4y^4 \\
&\qquad + \binom{8}{5}x^3y^5 + \binom{8}{6}x^2y^6 + \binom{8}{7}xy^7 + \binom{8}{8}y^8 \\
&= x^8 + 8x^7y + 28x^6y^2 + 56x^5y^3 + 70x^4y^4 + 56x^3y^5 + 28x^2y^6 + 8xy^7 + y^8.
\end{aligned}$$

. . . .

(b) Substitute $x = 2z$ and $y = 3$ to calculate $(2z + 3)^4$.

Answer. By the binomial theorem, we have

$$\begin{aligned}(2z + 3)^4 &= (2z)^4 + 4(2z)^3(3) + 6(2z)^2(3)^2 + 4(2z)(3)^3 + 3^4 \\ &= 2^4z^4 + 2^5 3^2 z^3 + 2^3 3^3 z^2 + 2^3 3^3 z + 3^4.\end{aligned}$$

. . . .

(c) What identity do you get if you substitute $x = -1$ and $y = 1$ in the binomial theorem? Check your identity (like in the previous problem) for $n = 4$.

Answer. Evaluating the binomial theorem at $x = 1$ and $y = -1$ says

$$\begin{aligned}0 = (1-1)^n = (x+y)^n\Big|_{\substack{x=1\\y=-1}} \\ &= \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}\Big|_{\substack{x=1\\y=-1}} \\ &= \sum_{k=0}^{n} \binom{n}{k} 1^k (-1)^{n-k} \\ &= \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \\ &= \sum_{k=0}^{n} (-1)^k \binom{n}{k} && \left(\text{Since } \binom{n}{k} = \binom{n}{n-k}\right)\end{aligned}$$

For example, when $n = 4$,

$$\sum_{k=0}^{4} (-1)^k \binom{4}{k} = \binom{4}{0} - \binom{4}{1} + \binom{4}{2} - \binom{4}{3} + \binom{4}{4} = 1 - 4 + 6 - 4 + 1 = 0 \quad \checkmark$$

. . . .

(d) Using the binomial theorem to prove combinatorial identities.

(i) Use the binomial theorem to explain why

$$2^n = \sum_{k=0}^{n} \binom{n}{k}$$

Then check and examples of this identity by calculating both sides for $n = 4$. (Hint: substitute $x = y = 1$).

Answer. Evaluating the binomial theorem at $x = y = 1$ says

$$\begin{aligned} 2^n = (1+1)^n &= (x+y)^n \Big|_{\substack{x=1\\y=1}} \\ &= \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \Bigg|_{\substack{x=1\\y=1}} \\ &= \sum_{k=0}^{n} \binom{n}{k} 1^k 1^{n-k} \\ &= \sum_{k=0}^{n} \binom{n}{k}. \end{aligned}$$

In particular, for $n = 4$,

$$\begin{aligned} \sum_{k=0}^{4} \binom{n}{k} &= \binom{4}{0} + \binom{4}{1} + \binom{4}{2} + \binom{4}{3} + \binom{4}{4} \\ &= 1 + 4 + 6 + 4 + 1 = 16 = 2^4 \quad \checkmark \end{aligned}$$

. . . .

(ii) Use the binomial theorem to explain why

$$2^n = (-1)^n \sum_{k=0}^{n} \binom{n}{k} (-3)^k.$$

Then check and examples of this identity by calculating both sides for $n = 4$. (Hint: what other examples can you think of of integers that sum to 2?).

Answer. Evaluating the binomial theorem at $x = 3$ and $y = -1$ says

$$\begin{aligned} 2^n = (3-1)^n &= (x+y)^n \Big|_{\substack{x=3\\y=-1}} \\ &= \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \Bigg|_{\substack{x=3\\y=-1}} \\ &= \sum_{k=0}^{n} \binom{n}{k} (3)^k (-1)^{n-k} \\ &= \sum_{k=0}^{n} \binom{n}{k} (-3)^k (-1)^n, \qquad \text{since } 3^k(-1)^{n-k} = 3^k(-1)^{-k}(-1)^n \\ &\qquad\qquad\qquad\qquad\qquad\qquad\quad = 3^k(-1)^k(-1)^n = (-3)^k(-1)^n, \\ &= (-1)^n \sum_{k=0}^{n} \binom{n}{k} (-3)^k. \end{aligned}$$

In particular, for $n = 4$,

$$\begin{aligned} (-1)^4 \sum_{k=0}^{4} \binom{n}{k} (-3)^k &= \binom{4}{0} + \binom{4}{1}(-3) + \binom{4}{2}(-3)^2 + \binom{4}{3}(-3)^3 + \binom{4}{4}(-3)^4 \\ &= 1 - 12 + 54 - 108 + 81 = 16 = 2^4 \quad \checkmark \end{aligned}$$

. . . .

(e) Give a formula for the coefficient of x^k in the expansion of $(x+1/x)^{100}$, where k is an integer.

Answer. Using the binomial theorem, evaluate at $y = 1/x = x^{-1}$. So the coefficient of x^k corresponds to $x^a(1/x)^{100-a}$ where $k = a-(100-a) = 2a-100$. So $a = (k+100)/2$. So x^k has coefficient 0 when k is odd or $\binom{100}{(k+100)/2}$ when k is even and between -100 and 100.

For a smaller example, consider

$$(x+1/x)^4 = x^4 + 4x^3(1/x) + 6x^2(1/x)^2 + 4x(1/x)^3 + (1/x)^4 = x^4 + 4x^2 + 6x^0 + 4x^{-2} + x^{-4}.$$

Check:

$$1 = \binom{4}{(4+4)/2}, \quad 4 = \binom{4}{(4+2)/2}, \quad 6 = \binom{4}{(4+0)/2}, \quad 4 = \binom{4}{(4-2)/2}, \quad 4 = \binom{4}{(4-4)/2}.$$

Alternatively, note that

$$(x+1/x)^{100} = \big((1/x)(x^2+1)\big)^{100} = \left(\frac{1}{x}\right)^{100}(x^2+1)^{100} = x^{-100}(x^2+1)^{100}.$$

By the binomial theorem,

$$(x^2+1)^{100} = \sum_{k=0}^{100}\binom{100}{k}(x^2)^k = \sum_{k=0}^{100}\binom{100}{k}x^{2k}.$$

So

$$(x+1/x)^{100} = x^{-100}\sum_{k=0}^{100}\binom{100}{k}x^{2k} = \sum_{k=0}^{100}\binom{100}{k}x^{2k-100}.$$

So the coefficient on x^ℓ is $\binom{100}{\frac{1}{2}(\ell+100)}$ when $\ell = 2k-100$ for $0 \le k \le 100$ and is 0 otherwise (which agrees with our solution above).

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(f) As a useful counting tool, we have (so far) only defined $\binom{n}{k}$ for non-negative integers n and k. But in §8.4 (p. 539, Def. 2), the book defines "extended binomial coefficients" as follows: Let $u \in \mathbb{R}$ and $k \in \mathbb{Z}_{\ge 0}$. The *extended binomial coefficient* $\binom{u}{k}$ is defined by

$$\binom{u}{k} = \begin{cases} u(u-1)(u-2)\cdots(u-k+1)/k! & \text{if } k > 0 \\ 1 & \text{if } k = 0. \end{cases}$$

For example,

$$\binom{0.2}{3} = \frac{0.2(-0.8)(-1.8)}{3*2*1}.$$

(i) Compute $\binom{\pi}{4}$, $\binom{1/2}{2}$, and $\binom{7/3}{0}$.

Answer. We have

$$\binom{\pi}{4} = \pi(\pi-1)(\pi-2)(\pi-3)/4! \approx 4.045;$$
$$\binom{1/2}{2} = (1/2)(1/2-1)/2! = -1/8; \text{ and}$$
$$\binom{7/3}{0} = 1, \text{ by definition, since } k = 0 \text{ here.}$$

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(ii) Verify algebraically that if n is a positive integer and $0 \le k \le n$, then

$$\binom{-n}{k} = (-1)^k \binom{n+k-1}{k}.$$

Check this identity for $n = 5$ and $k = 3$ (you should probably do this first).

Proof. For $k = 0$, then both sides are equal to 1. Otherwise, we have

$$\begin{aligned}
\binom{-n}{k} &= \frac{(-n)(-n-1)(-n-2)\cdots(-n-k+1)}{k!} \\
&= (-1)^k \frac{n(n+1)(n+2)\cdots(n+k-1)}{k!} \\
&= (-1)^k \frac{X(X-1)(X-2)\cdots\big(X-k+1\big)}{k!}, \\
&\qquad \text{where } X = n+k-1, \text{ flipping the order of multiplication,} \\
&\qquad \text{in particular, since } X-k+1 = (n+k-1)-k+1 = n, \\
&= (-1)^k \binom{n+k-1}{k},
\end{aligned}$$

as desired. □

(iii) BONUS: The *extended binomial theorem* states that for any real number u, we have

$$(1+x)^u = \sum_{i=0}^{\infty} \binom{u}{k} x^k.$$

Now recall from calculus the Taylor series expansion

$$(1+x)^{-1} = \sum_{i=0}^{\infty} x^i(-1)^i.$$

Check that the first 3 terms ($i = 0, 1, 2$) of our known Taylor series expansion match the first 3 terms of the extended binomial theorem expansion (when $u = -1$). Finally, verify that this example matches correctly for all terms by showing that $\binom{-1}{k} = (-1)^k$ for any $k \in \mathbb{Z}_{\ge 0}$.

Proof. Note that

$$\begin{aligned}
\binom{-1}{k} &= \frac{-1(-1-1)(-1-2)\cdots(-1-k+1)}{k!} \\
&= \frac{-1(-2)(-3)\cdots(-k)}{k!} \\
&= \frac{(-1)^k k!}{k!} = (-1)^k.
\end{aligned}$$

So the extended binomial theorem says that

$$(1+x)^{-1} = \sum_{i=0}^{\infty} \binom{-1}{k} x^k = \sum_{i=0}^{\infty} (-1)^k x^k,$$

as desired. □

(iv) BONUS: Show in that the Taylor series expansion for $(1+x)^{-n}$ matches the extended binomial theorem for $n = 3$.

Proof. We have

$$\begin{aligned}
\binom{-3}{k} &= \frac{-3(-3-1)(-3-2)\cdots(-3-k+1)}{k!} \\
&= \frac{-3(-4)(-5)\cdots(-k)(-(k+1))(-(k+2))}{k!} \\
&= \frac{(-1)(-2)(-3)(-4)(-5)\cdots(-k)(-(k+1))(-(k+2))}{(-1)(-2)k!} \\
&= \frac{(-1)^{k+2}k!(k+1)(k+2)}{2(k!)} \\
&= (-1)^{k+2}\frac{(k+1)(k+2)}{2} \\
&= (-1)^{k}\frac{(k+1)(k+2)}{2}.
\end{aligned}$$

So the extended binomial theorem says that

$$(1+x)^{-3} = \sum_{i=0}^{\infty}\binom{-3}{k}x^k = \sum_{i=0}^{\infty}(-1)^k\frac{(k+1)(k+2)}{2}x^k = \sum_{i=0}^{\infty}\frac{(k+1)(k+2)}{2}(-x)^k.$$

On the other hand, we have

$$\begin{aligned}
\frac{d}{dx}(1+x)^{-1} &= -(1+x)^-2; \quad \text{so that} \\
\frac{d^2}{dx^2}(1+x)^{-1} &= \frac{d}{dx}\left(-(1+x)^-2\right) = 2(1+x)^{-3}.
\end{aligned}$$

Thus

$$\begin{aligned}
(1+x)^{-3} &= \frac{1}{2}\frac{d^2}{dx^2}(1+x)^{-1} \\
&= \frac{1}{2}\frac{d^2}{dx^2}\left(\sum_{k=0}^{\infty}(-1)^k x^k\right) \\
&= \frac{1}{2}\sum_{k=0}^{\infty}(-1)^k\left(\frac{d^2}{dx^2}x^k\right) \\
&= \frac{1}{2}\sum_{k=2}^{\infty}(-1)^k\left(k(k-1)x^{k-2}\right) \\
&= \frac{1}{2}\sum_{k=0}^{\infty}(-1)^k(k+2)(k+1)x^k \\
&= \frac{1}{2}\sum_{k=0}^{\infty}\frac{(k+1)(k+2)}{2}(-x)^k,
\end{aligned}$$

which matches what we got from the extended binomial theorem. □

Exercise 25.

(a) Explain the example provided for the proof of Vandermonde's in the notes using words.

Answer. In proving Vandermonde's identity, we showed that you can count the size r subsets of $A \sqcup B$ in two ways, where $|A| = n$, $|B| = m$, and $r \leq n, m$. The first way was to count them all at once. In this example, when $r = 3$, these are the subsets listed in the table. The second way to count them is take size k subsets of A and size $r - k$ subsets of B and union them. These correspond to the different columns of the table (the first column is taking no elements from B, the second column is taking one element from B, and so on.

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(b) Substitute $m = r = n$ into Vandermonde's identity to show that

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2,$$

and check this identity for $n = 2$.

Answer. We have

$$\begin{aligned} \binom{n+m}{r}\Big|_{m=r=n} &= \binom{2n}{n} \\ &= \sum_{j=0}^{r} \binom{m}{j}\binom{n}{r-j}\Big|_{m=r=n} \\ &= \sum_{j=0}^{n} \binom{n}{j}\binom{n}{n-j} = \sum_{j=0}^{n} \binom{n}{j}^2. \end{aligned}$$

For $n = 2$, we get

$$\begin{aligned} \binom{2*2}{2} &= \binom{4}{2} = 6 \\ = \sum_{j=0}^{2} \binom{2}{j}^2 &= \binom{2}{0}^2 + \binom{2}{1}^2 + \binom{2}{2}^2 \\ &= 1^2 + 2^2 + 1^2 = 6. \quad \checkmark \end{aligned}$$

. . . .

(c) Consider the identity

$$\binom{n}{k} k = \binom{n-1}{k-1} n$$

for integers $1 \leq k \leq n$.

(i) Verify this identity for $n = 5$ and $k = 3$.

Answer. We have

$$\begin{aligned} \binom{5}{3} 3 &= 10 * 3 = 30 \\ &= \binom{5-1}{3-1} 5 = \binom{4}{2} 5 = 6 * 5 = 30. \qquad \checkmark \end{aligned}$$

. . . .

(ii) Explain why this identity is true using a combinatorial argument.
[*Hint: Count, in two different ways, the number of ways to pick a subset with k elements from a set with n elements, along with a distinguished element of that k-element subset. For example, out of n people, pick a committee of k people and choose someone on that committee to organize their meetings.*]

Answer. We can count the number of ways to choose a committee of k people from n and a person to run that committee in two ways. First, choose the committee and then someone to run it from amongst those people. There are

$$\binom{n}{k}k \qquad \text{ways to do this.}$$

On the other hand, you could fist choose the person to run the committee, and then choose the other $k-1$ members from the remaining $n-1$ people. There are

$$\binom{n-1}{k-1}n \qquad \text{ways to do this.}$$

Since $\binom{n}{k}k$ and $\binom{n-1}{k-1}n$ are two different ways of expressing the size of the same set, they must be equal.

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(iii) Illustrate your combinatorial proof using the set $A = \{a, b, c\}$ (so that $n = 3$) and $k = 2$.

Answer. In the first way of counting, we have

committee $\{a, b\}$ with leader a,
committee $\{a, b\}$ with leader b,
committee $\{a, c\}$ with leader a,
committee $\{a, c\}$ with leader c,
committee $\{b, c\}$ with leader b, and
committee $\{b, c\}$ with leader c.

In the second way of counting, we have

leader a with other member b,
leader b with other member a,
leader a with other member c,
leader c with other member a,
leader b with other member c, and
leader c with other member b.

Either way, there are 6 choices.

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(iv) Verify the identity algebraically using the formula $\binom{n}{k} = n!/((n-k)!k!)$.

Answer. Algebraically, we have

$$\binom{n}{k}k = \frac{n!}{((n-k)!k!}k = \frac{n!}{((n-k)!(k-1)!} = \frac{n*(n-1)!}{((n-1)-(k-1))!(k-1)!} = n\binom{n-1}{k-1}.$$

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