SOLUTIONS
Homework 3
Due 2/20/19

Exercise 15. Show that each of the following sets are countably infinite by giving a bijective function between that set and the positive integers.

(a) the integers greater than 10.

Answer. The function

$$\begin{array}{ccccc} 11 & 12 & 13 & 14 & \cdots \\ \downarrow & \downarrow & \downarrow & \downarrow & \\ 0 & 1 & 2 & 3 & \cdots \end{array}$$

$$\begin{aligned} f : \mathbb{Z}_{>10} &\to \mathbb{Z}_{\geq 0} \\ z &\mapsto z - 11 \end{aligned}$$

is bijective.

....

(b) the odd negative integers

Answer. The function

$$\begin{array}{cccc} 1 & 2 & 3 & \cdots \\ \downarrow & \downarrow & \downarrow & \\ -1 & -3 & -5 & \cdots \end{array}$$

$$\begin{aligned} f : \mathbb{Z}_{\geq 0} &\to \{n \in \mathbb{Z}_{<0} \mid n \text{ is odd }\} \\ x &\mapsto -(2x + 1) \end{aligned}$$

is bijective.

....

(c) the set $A \times \mathbb{Z}^+$, where $A = \{2, 3\}$

Answer. The function

$$\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & \cdots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ (2,1) & (3,1) & (2,2) & (3,2) & (2,3) & (3,3) & \cdots \end{array}$$

$$\begin{aligned} f : \mathbb{Z}_{\geq 0} &\to A \times \mathbb{Z}^+ \\ x &\mapsto \begin{cases} (2, (x+1)/2) & \text{if } x \text{ is odd,} \\ (3, x/2) & \text{if } x \text{ is even.} \end{cases} \end{aligned}$$

is bijective.

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(d) the integers that are multiples of 10

Answer. The function

$$\begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 5 & \cdots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ 0 & -10 & 10 & -20 & 20 & -30 & \cdots \end{array}$$

$$f : \mathbb{Z}_{\geq 0} \to 10\mathbb{Z}$$
$$x \mapsto \begin{cases} 5x & \text{if } x \text{ is even,} \\ -5(x+1) & \text{if } x \text{ is odd.} \end{cases}$$

is bijective.

. . . .

Exercise 16.

(a) Determine whether each of these sets is finite, countable, or uncountable. For those that are countably infinite, exhibit a bijective correspondence between the set of positive integers and that set.

(i) The integers that are multiples of 10.

Answer. See 15 (d).

. . . .

(ii) Integers not divisible by 3.

Answer. This set is countable. Namely, the integers are listable, and we can build a list from that list by simply skipping all the multiples of 3:

$$\begin{array}{ccccccccccccccccc} & 1 & 2 & 3 & 4 & & & 5 & 6 & 7 & 8 & & & 9 & 10 & \cdots \\ & \downarrow & \downarrow & \downarrow & \downarrow & & & \downarrow & \downarrow & \downarrow & \downarrow & & & \downarrow & \downarrow & \\ 0 & -1 & 1 & -2 & 2 & -3 & 3 & -4 & 4 & -5 & 5 & -6 & 6 & -7 & 7 & \cdots \end{array}$$

. . . .

(iii) The real numbers with decimal representations consisting of all 1s.

Answer. This set is countable as follows:

$$\begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & \cdots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ 0.\overline{1} & -1.\overline{1} & 1.\overline{1} & -2.\overline{1} & 2.\overline{1} & -3.\overline{1} & 3.\overline{1} & \cdots \end{array}$$

. . . .

(iv) The real numbers with decimal representations of all 1s or 9s.

Answer. This set is not countable. In particular, let A be the set of real numbers with decimal representations of all 1s or 9s. And suppose you have a list of numbers $a_1, a_2, a_3, \ldots$ in A. Then we can build a number that's not one of those a_i's as follows: for any $x \in \mathbb{R}$, let $x^{(j)}$ be the jth decimal of x; for example, if

$$a_i = 37.1191\ldots,$$

then

$$a_i^{(1)} = 1, \quad a_i^{(2)} = 1, \quad a_i^{(3)} = 1, \quad a_i^{(4)} = 1,$$

and so on. Then let $a = 0.a^{(1)}a^{(2)}a^{(3)}a^{(4)}\ldots$, where

$$a^{(j)} = \begin{cases} 1 & \text{if } a_j^{(j)} = 9, \\ 9 & \text{if } a_j^{(j)} = 1. \end{cases}$$

So, by definition

$$\begin{aligned} a \neq a_1 \quad &\text{since} \quad a^{(1)} \neq a_1^{(1)}; \\ a \neq a_2 \quad &\text{since} \quad a^{(2)} \neq a_1^{(2)}; \\ a \neq a_3 \quad &\text{since} \quad a^{(3)} \neq a_1^{(3)}; \\ a \neq a_4 \quad &\text{since} \quad a^{(4)} \neq a_1^{(4)}; \\ &\vdots \end{aligned}$$

and so on. Thus a is in A but isn't on the list. Therefore the elements of A aren't listable.

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(v) The integers with absolute value less than $1,000$.

Answer. This is a finite set:

$$\{x \in \mathbb{Z} \mid |X| < 1,000\} = \{-999, -998, \ldots, 997, 998, 999\},$$

which has $1,999$ elements.

. . . .

(vi) The real numbers between 0 and 2

Answer. This set is not countable. In particular, the functions

$$\begin{aligned} f : [0,2] &\to \mathbb{R} \\ x &\mapsto x \end{aligned} \qquad \text{and} \qquad \begin{aligned} g : \mathbb{R} &\to [0,2] \\ x &\mapsto \tfrac{2}{\pi}\arctan(x) + 1 \end{aligned}$$

are both injective functions, so $|[0,2]| = |\mathbb{R}|$.

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(b) Give an example of two uncountable sets A and B such that $A - B$ is

(i) finite; *Answer:* $A = B = \mathbb{R}$, so $|A - B| = |\emptyset| = 0$

(ii) countably infinite;

Answer. By similar reasoning as (a) part (vi), we can show that the interval $(0, 1)$ is uncountable, and therefore $B = \bigcup_{i\in\mathbb{Z}}(i, i + 1)$ is uncountable. So if $A = \mathbb{R}$, then both A and B are uncountable, but $A - B = \mathbb{Z}$, which is countably infinite.

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(iii) uncountable. *Answer:* $A = \mathbb{R}$, $B = \mathbb{R}_{<0}$, so $|A - B| = \mathbb{R}_{\geq 0}$.

(c) Explain why the power set of $\mathbb{Z}_{\geq 1}$ is not countable as follows:

(i) First, for each subset $A \subset \mathbb{Z}_{\geq 1}$, represent A as an infinite bit string (a sequence of 1's and 0's with no end to the right) with ith bit 1 if i belongs to the subset and 0 otherwise. Give the bit-string expansions for the sets $\{2, 4, 6, 7\}$ and $\{3x \mid x \in \mathbb{Z}_{\geq 1}\}$ (i.e. the positive multiples of 3); and give the set corresponding to the bitstring expansions 0000000000000. . . and 111111111. . . . Finally, explain why this coding of sets as bit strings is actually a bijection between {infinite bit strings} and $\mathcal{P}(\mathbb{Z}_{\geq 1})$.

Answer.

$$\begin{aligned} \{2,4,6,7\} &\leftrightarrow 010101100000\ldots \\ \{3x \mid x \in \mathbb{Z}_{\geq 1}\} &\leftrightarrow 001001001001\ldots \\ \emptyset &\leftrightarrow 000000000000\ldots \\ \mathbb{Z}_{\geq 1} &\leftrightarrow 111111111111\ldots \end{aligned}$$

This coding of sets as bit strings is a bijection between {infinite bit strings} and $\mathcal{P}(\mathbb{Z}_{\geq 1})$ because it gives an invertible function from $\mathcal{P}(\mathbb{Z}_{\geq 1})$ to {infinite bit strings}.

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(ii) Suppose that you can list these infinite strings in a list labeled by the positive integers (as we saw, this is the same as saying that there is some bijective map f : {infinite bit strings} $\to \mathbb{Z}_{\geq 1}$). Construct a new bit string one bit at a time, so that it doesn't match the ith string in the ith bit. Conclude that your new string can't be in the list, so that the list wasn't actually complete.

Answer. Call the ith bit-string s_i, and let the jth terms in a string s be denoted by $s^{(j)}$. Then define the string s by

$$s^{(j)} = \begin{cases} 1 & \text{if } s_j^{(j)} = 0, \\ 0 & \text{if } s_j^{(j)} = 1. \end{cases}$$

Then since $s^{(j)} \neq s_j^{(j)}$ for all j, we know $s \neq s_j$ for all j. So s is not on the list.

. . . .

(iii) Finally, explain how to use (i) and (ii) together to show that the sets in $\mathcal{P}(\mathbb{Z}_{\geq 1})$ aren't listable (and therefore aren't countable).

Answer. Since every list with elements in $\mathcal{P}(\mathbb{Z}_{\geq 1})$ is incomplete, the elements of $\mathcal{P}(\mathbb{Z}_{\geq 1})$ are not totally listable. Thus $\mathcal{P}(\mathbb{Z}_{\geq 1})$ isn't countable.

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(d) Show that if A and B are sets and $A \subset B$, then $|A| \leq |B|$.
[Hint: Start with thinking about the definition of what it means for $|A| \leq |B|$.]

Proof. We define $|A| \leq |B|$ by the existence of an injective function $f : A \to B$. But if $A \subseteq B$, then $f : A \to B$ defined by $a \mapsto a$ is a well-defined injective function. So $|A| \leq |B|$. □

(e) Show that a subset of a countable set is also countable.

Proof. Suppose A is a countable set, and that $B \subseteq A$. Since A is countable, there is a bijective function $f : A \to \mathbb{Z}_{\geq 1}$. But then $f|_B : B \to \mathbb{Z}_{\geq 1}$ is an injective function so that $|B| \leq \aleph_0$. So B is either countable or is finite. □

(f) Use the Schröder-Bernstein theorem to show that $(0, 1)$ and $[0, 1]$ have the same cardinality.

Proof. The functions

$$\begin{aligned} f : (0,1) &\to [0,1] \\ x &\mapsto x \end{aligned} \qquad \text{and} \qquad \begin{aligned} g : [0,1] &\to (0,1) \\ x &\mapsto .1 + x/2 \end{aligned}$$

are both injective functions. So $|(0, 1)| \leq |[0, 1]|$ and $|(0, 1)| \geq |[0, 1]|$, and thus by the Schröder-Bernstein theorem, $|(0, 1)| = |[0, 1]|$. □

Exercise 17. For each of the following, outline a proof by induction. Where indicated, also write a "final draft" version of your proof.

(a) Show for $n \in \mathbb{N}$ and $r \neq 1$

$$\sum_{i=0}^{n} r^i = \frac{r^{n+1}-1}{r-1}.$$

Also give a final draft proof.

Outline proof.

- Define $P(n)$: $\sum_{i=0}^{n} r^i = \frac{r^{n+1}-1}{r-1}$.
- Base case: For $n = 0$, we have

$$\sum_{i=0}^{0} r^i = r^0 = 1 = \frac{r^1-1}{r-1}. \quad \checkmark$$

- Write down $P(n+1)$:

$$\sum_{i=0}^{n+1} r^i = \frac{r^{n+2}-1}{r-1}.$$

- Inductive step:
  Fix $n \geq 0$ and assume $\sum_{i=0}^{n} r^i = \frac{r^{n+1}-1}{r-1}$. Then

$$\begin{aligned}\sum_{i=0}^{n+1} r^i = r^{n+1} + \sum_{i=0}^{n} r^i &\overset{\text{IH}}{=} r^{n+1} + \frac{r^{n+1}-1}{r-1} \\ &= \frac{r^{n+2}-r^{n+1}+r^{n+1}-1}{r-1} \\ &= \frac{r^{n+2}-1}{r-1}. \quad \checkmark\end{aligned}$$

- Make a conclusion:
  Since $P(0)$ holds and $P(n)$ implies $P(n+1)$ for all $n \geq 0$, we have $P(n)$ holds for all $n \geq 0$.

□

Final draft proof. First, for $n = 0$, we have

$$\sum_{i=0}^{0} r^i = r^0 = 1 = \frac{r^1-1}{r-1}.$$

Next, fix $n \geq 0$ and assume $P(n)$ holds. Then

$$\begin{aligned}\sum_{i=0}^{n+1} r^i = r^{n+1} + \sum_{i=0}^{n} r^i &= r^{n+1} + \frac{r^{n+1}-1}{r-1}, \quad \text{by the inductive hypothesis,} \\ &= \frac{r^{n+2}-r^{n+1}+r^{n+1}-1}{r-1} \\ &= \frac{r^{n+2}-1}{r-1},\end{aligned}$$

as desired. Therefore, the claim holds for all $n \geq 0$ by induction. □

(b) For $n = 1, 2, 3$, calculate $\sum_{i=1}^{n} 2i - 1$ (the sum of the first n odd numbers). Notice that in each case, $\displaystyle\sum_{i=1}^{n} 2i - 1 = n^2$. Show that this is true in general.

Answer. We have

$$\sum_{i=1}^{1} 2i - 1 = 2 * 1 - 1 = 1,$$
$$\sum_{i=1}^{2} 2i - 1 = (2 * 1 - 1) + (2 * 2 - 1) = 1 + 3 = 4, \text{ and}$$
$$\sum_{i=1}^{3} 2i - 1 = (2 * 1 - 1) + (2 * 2 - 1) + (2 * 3 - 1) = 1 + 3 + 5 = 9.$$

. . . .

Outline proof.

- Define $P(n)$: $\sum_{i=1}^{n} 2i - 1 = n^2$.
- Base case: For $n = 1$, we have

$$\sum_{i=1}^{1} 2i - 1 = 2 * 1 - 1 = 1 = 1^2. \quad \checkmark$$

- Write down $P(n + 1)$:

$$\sum_{i=1}^{n+1} 2i - 1 = (n + 1)^2.$$

- Inductive step:
  Fix $n \geq 1$ and assume $\sum_{i=1}^{n} 2i - 1 = n^2$. Then

$$\begin{aligned} \sum_{i=1}^{n+1} 2i - 1 &= 2(n + 1) - 1 + \sum_{i=1}^{n} 2i - 1 \\ &\overset{\text{IH}}{=} 2(n + 1) - 1 + n^2 \\ &= n^2 + 2n + 2 - 1 \\ &= (n + 1)^2. \quad \checkmark \end{aligned}$$

- Make a conclusion:
  Since $P(1)$ holds and $P(n)$ implies $P(n + 1)$ for all $n \geq 1$, we have $P(n)$ holds for all $n \geq 1$.

$\square$

(c) Show $n^3 + 2n$ is a multiple of 3 for all $n \in \mathbb{N}$. **Also give a final draft proof.**

Outline proof.

- Define $P(n)$: $n^3 + 2n = 3k$ for some $k \in \mathbb{Z}$.
- Base case: For $n = 0$, we have $0^3 + 2 \cdot 0 = 0 = 3 \cdot 0$. ✓
- Write down $P(n+1)$:

$$(n+1)^3 + 2(n+1) = 3\ell \quad \text{for some } \ell \in \mathbb{Z}.$$

- Inductive step:
  Fix $n \geq 0$ and assume $n^3 + 2n = 3k$ for some $k \in \mathbb{Z}$. Then

$$(n+1)^3 + 2(n+1) = n^3 + 3n^2 + 3n + 1 + 2n + 2 = (n^3 + 2n) + 3(n^2 + n + 1)$$
$$\stackrel{\text{IH}}{=} 3k + 3(n^2 + n + 1) = 3(k + n^2 + n + 1).$$

  Thus, since $k + n^2 + n + 1 \in \mathbb{Z}$, we have $(n+1)^3 + 2(n+1)$ is a multiple of 3 as well.
- Make a conclusion:
  Since $P(0)$ holds and $P(n)$ implies $P(n+1)$ for all $n \geq 0$, we have $P(n)$ holds for all $n \geq 0$.

□

Final draft proof. We will prove this claim by induction on n. First, when $n = 0$, we have

$$0^3 + 2 \cdot 0 = 0 = 3 \cdot 0,$$

as desired. Next, fix $n \geq 0$ and assume $n^3 + 2n = 3k$ for some $k \in \mathbb{Z}$. Then

$$\begin{aligned}(n+1)^3 + 2(n+1) &= n^3 + 3n^2 + 3n + 1 + 2n + 2\\ &= (n^3 + 2n) + 3(n^2 + n + 1)\\ &= 3k + 3(n^2 + n + 1), \quad \text{by the induction hypothesis,}\\ &= 3(k + n^2 + n + 1).\end{aligned}$$

Thus, since $k + n^2 + n + 1 \in \mathbb{Z}$, we have $(n+1)^3 + 2(n+1)$ is a multiple of 3 as well.

Therefore, the claim holds for all $n \geq 0$ by induction. □

(d) Show $n! < n^n$ for $n > 1$.

Outline proof.

- Define $P(n)$: $n! < n^n$.
- Base case: For $n = 2$, we have

$$2! = 2 < 4 = 2^2. \quad ✓$$

- Write down $P(n+1)$:

$$(n+1)! < (n+1)^{n+1}.$$

- Inductive step:
  Fix $n \geq 2$ and assume $n! < n^n$. Then

$$\begin{aligned}(n+1)! &= (n+1)(n!)\\ &\stackrel{\text{IH}}{<} (n+1)n^n\\ &< (n+1)(n+1)^n = (n+1)^{n+1}. \quad ✓\end{aligned}$$

- Make a conclusion:
  Since $P(2)$ holds and $P(n)$ implies $P(n+1)$ for all $n \geq 2$, we have $P(n)$ holds for all $n \geq 2$.

□

(e) Suppose $A_1, A_2, \dots A_N$ and $B_1, B_2, \dots, B_N$ are sets such that $A_i \subseteq B_i$ for all $1 \le i \le N$. Then

$$\bigcup_{i=1}^{N} A_i \subseteq \bigcup_{i=1}^{N} B_i.$$

Also give a final draft proof.

Outline proof.

- Define $P(N)$: If $A_1, A_2, \dots A_N$ and $B_1, B_2, \dots, B_N$ are sets such that $A_i \subseteq B_i$ for all $1 \le i \le N$, then

$$\bigcup_{i=1}^{N} A_i \subseteq \bigcup_{i=1}^{N} B_i.$$

- Base case: For $n = 1$, we have

$$\text{if } A_1 \subseteq B_1 \quad \text{then} \quad \bigcup_{i=1}^{1} A_i \subseteq \bigcup_{i=1}^{1} B_i. \quad \checkmark$$

  We will also need $P(2)$, which is "if $A_1 \subseteq B_1$ and $A_2 \subseteq B_2$ then $A_1 \cup A_2 \subseteq B_1 \cup B_2$."

  Proof. If $x \in A_1 \cup A_2$, then $x \in A_1$ or $x \in A_2$. But since $A_1 \subseteq B_1$ and $A_2 \subseteq B_2$, this implies $x \in B_1$ or $x \in B_2$. So $x \in B_1 \cup B_2$. Thus $A_1 \cup A_2 \subseteq B_1 \cup B_2$. □

- Write down $P(n+1)$: If $A_1, A_2, \dots A_{N+1}$ and $B_1, B_2, \dots, B_{N+1}$ are sets such that $A_i \subseteq B_i$ for all $1 \le i \le N+1$, then

$$\bigcup_{i=1}^{N+1} A_i \subseteq \bigcup_{i=1}^{N+1} B_i.$$

- Inductive step:
  Fix $N \ge 2$ and assume that for any sets $A_1, A_2, \dots A_N$ and $B_1, B_2, \dots, B_N$ such that $A_i \subseteq B_i$ for all $1 \le i \le N$, we have

$$\bigcup_{i=1}^{N} A_i \subseteq \bigcup_{i=1}^{N} B_i.$$

  . Then

$$\bigcup_{i=1}^{N+1} A_i = \left(\bigcup_{i=1}^{N} A_i\right) \cup A_{N+1}.$$

  By the inductive hypothesis, we know $\bigcup_{i=1}^{N} A_i \subseteq \bigcup_{i=1}^{N} B_i$. And by $P(2)$ (which we proved above), since we also have $A_{N+1} \subseteq B_{N+1}$, we can conclude that

$$\left(\bigcup_{i=1}^{N} A_i\right) \cup A_{N+1} \subseteq \left(\bigcup_{i=1}^{N} B_i\right) \cup B_{N+1}.$$

  Therefore

$$\bigcup_{i=1}^{N} A_i \subseteq \bigcup_{i=1}^{N} B_i,$$

  as desired.

- Make a conclusion:
  Since $P(1)$ and $P(2)$ hold and $P(N)$ implies $P(N+1)$ for all $N \ge 2$, we have $P(N)$ holds for all $N \ge 1$.

□

(f) Suppose $A_1, A_2, \dots A_N$ and B are sets. Then

$$(A_1 - B) \cap (A_2 - B) \cap \cdots \cap (A_N - B) = (A_1 \cap A_2 \cap \cdots \cap A_N) - B.$$

Outline proof.

- Define $P(N)$: If $A_1, A_2, \dots A_N$ and B are sets, then

$$(A_1 - B) \cap (A_2 - B) \cap \cdots \cap (A_N - B) = (A_1 \cap A_2 \cap \cdots \cap A_N) - B.$$

- Base case: For $n = 1$, we have $A_1 - B = A_1 - B$. ✓ We will also need $P(2)$, which is "if A_1, A_2, and B are sets, then $(A_1 - B) \cap (A_2 - B) = (A_1 \cap A_2) - B$."

  Proof. If $x \in (A_1 - B) \cap (A_2 - B)$, then $x \in A_1 - B$, so that $x \in A_1$ but not B; and $x \in A_2 - B$, so that $x \in A_1$ but not B. So x is in A_1 and A_2 but not B, i.e. $x \in (A_1 \cap A_2) - B$.
  Conversely, if $x \in (A_1 \cap A_2) - B$, then $x \in A_1$ and $x \in A_2$ but $x \notin B$. So $x \in A_1 - B$ and $x \in A_2 - B$. Thus $x \in (A_1 - B) \cap (A_2 - B)$.
  Therefore $(A_1 - B) \cap (A_2 - B) = (A_1 \cap A_2) - B$, as desired. □

- Write down $P(n + 1)$:If $A_1, A_2, \dots A_{N+1}$ and B are sets, then

$$(A_1 - B) \cap (A_2 - B) \cap \cdots \cap (A_{N+1} - B) = (A_1 \cap A_2 \cap \cdots \cap A_{N+1}) - B.$$

- Inductive step:
  Fix $N \geq 2$ and assume that for any sets $A_1, A_2, \dots A_N$ and B are sets, then

$$(A_1 - B) \cap (A_2 - B) \cap \cdots \cap (A_N - B) = (A_1 \cap A_2 \cap \cdots \cap A_N) - B.$$

  Then

$$\begin{aligned}&\Big((A_1 - B) \cap (A_2 - B) \cap \cdots \cap (A_N - B)\Big) \cap (A_{N+1} - B)\\ &\overset{\text{IH}}{=} \Big((A_1 \cap A_2 \cap \cdots \cap A_N) - B\Big) \cap (A_{N+1} - B)\\ &\overset{P(2)}{=} (A_1 \cap A_2 \cap \cdots \cap A_{N+1}) - B,\end{aligned}$$

  as desired.

- Make a conclusion:
  Since $P(1)$ and $P(2)$ hold and $P(N)$ implies $P(N + 1)$ for all $N \geq 2$, we have $P(N)$ holds for all $N \geq 1$.

□

Exercise 18.

(a) Suppose that you know that a golfer plays the first hole of a golf course with an infinite number of holes and that if this golfer plays one hole, then the golfer goes on to play the next hole. Use "proof by induction" reasoning to explain why this golfer must play every hole on the course.

Proof. Let $P(n)$ be the statement "This golfer can play the nth hole on the course." First, for $n = 1$, we have $P(1)$ is "this golfer can play the 1st hole on the course", which holds. Next, fix $n \geq 1$ and assume that this this golfer can play the nth hole on the course. Then since if this golfer plays one hole, then the golfer goes on to play the next hole, we know $P(n)$ implies $P(n+1)$. Thus, since $P(1)$ holds and $P(n)$ implies $P(n+1)$ for all $n \geq 1$, we have $P(n)$ holds for all $n \geq 1$. □

(b) Find the error in the following faulty proof that all horses are the same color.

Proof. Let $P(n)$ be the proposition that all the horses in a set of n horses are the same color. First, $P(1)$ is true since in any group of 1 horse, all the horses must be the same color. Now, fix $n \geq 1$ and assume that $P(n)$ is true, so that all the horses in any set of n horses are the same color. Consider any $n+1$ horses; number these as horses $1, 2, 3, \ldots, n, n+1$. Now **the first n of these horses all must have the same color, and the last n of these must also have the same color**. Because the set of the first n horses and the set of the last n horses overlap, all $n+1$ must be the same color. Thus $P(n)$ is true for all $n \geq 1$. □

Answer. The highlighted implication above breaks at $n = 1$, since there's no overlap between the first 1 horse and the last one horse.

. . . .

(c) The following is an example of why checking the base case is important.

Let $P(n)$ be the statement $\displaystyle\sum_{i=1}^{n} i = \frac{\left(n + \frac{1}{2}\right)^2}{2}$.

(i) Show that $P(n)$ implies $P(n+1)$. Namely, assume $\sum_{i=1}^{n} i = \frac{\left(n+\frac{1}{2}\right)^2}{2}$ for some $n \geq 1$, and <u>use that to show</u> that $\sum_{i=1}^{n+1} i = \frac{\left((n+1)+\frac{1}{2}\right)^2}{2}$.

Proof. Suppose $\sum_{i=1}^{n} i = \frac{\left(n+\frac{1}{2}\right)^2}{2}$ for some fixed n. Then

$$\begin{aligned}
\sum_{i=1}^{n+1} i &= (n+1) + \sum_{i=1}^{n} i \\
&\overset{\text{IH}}{=} (n+1) + \frac{\left(n + \frac{1}{2}\right)^2}{2} \\
&= \frac{1}{2}\left(2n + 2 + n^2 + n + \frac{1}{4}\right) \\
&= \frac{1}{2}\left(n^2 + 3n + \frac{9}{4}\right) \\
&= \frac{\left((n+1) + \frac{1}{2}\right)^2}{2}. \quad \checkmark
\end{aligned}$$

□

(ii) Check the base case, $P(1)$.

Answer. We have $\sum_{i=1}^{1} i = 1$, but $\frac{\left(1+\frac{1}{2}\right)^2}{2} = 9/8 \neq 1$.

. . . .

(iii) Prove that $P(n)$ is actually false for all $n \in \mathbb{Z}_{\geq 1}$.

Proof. Recall that we proved $\sum_{i=1}^{n} i = n(n+1)/2$. But if we also had $\sum_{i=1}^{n} i = \frac{\left(n+\frac{1}{2}\right)^2}{2}$ for *any* n, then

$$\frac{n(n+1)}{2} = \frac{\left(n+\frac{1}{2}\right)^2}{2}.$$

Expanding both sides and multiplying by 2 gives

$$n^2 + n = n^2 + n + \frac{1}{4}.$$

Subtracting $n^2 + n$ from both sides implies $0 = 1/4$, which is clearly wrong. Therefore $\sum_{i=1}^{n} i \neq \frac{\left(n+\frac{1}{2}\right)^2}{2}$ for all n.

□

Attach at the end of Homework 3:

At the end of your write-up, include the following, labeling this as **"Writing exercise"**.

(a) Mark up your finished homework assignment, showing where you followed or failed to follow the mechanical and stylistic issues outlined in the handout *Communicating Mathematics through Homework and Exams.* This means **treat your write-up as a second-to-last draft**, and go point-by-point through the handout and address instances where you followed or did not follow each direction in your writing. Use a different-colored pen if you have one, and hand in this marked up draft. You do not need to rewrite the result.

How did you improve this week over homework 1? How might you improve in the future?

(b) List three or more ways that you succeeded or failed at following the advice in *Some Guidelines for Good Mathematical Writing.* How did you improve this week over homework 1? How might you improve in the future?

To receive credit for this assignment, you must complete this exercise. **To receive any credit for homework 2, you must do this writing exercise.**