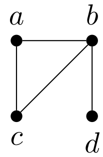
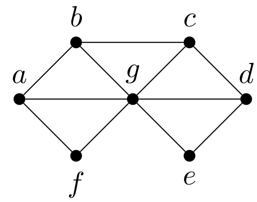
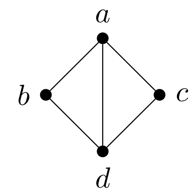
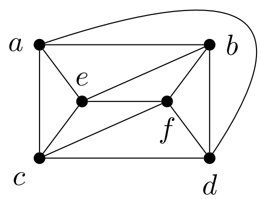
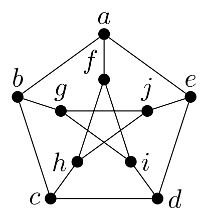
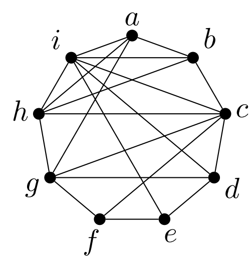
SOLUTIONS
Homework 11
Due 5/10/19

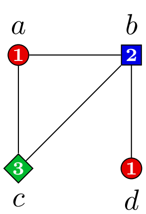
Exercise 55. Let A, B, C, D, G, and H be the graphs

$A =$ 
$B =$ 
$C =$ 

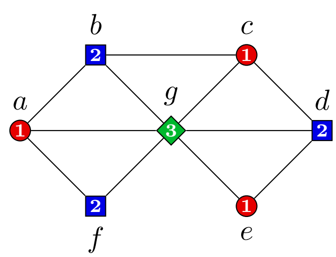
$D =$ 
$G =$ 
$H =$ 

(a) Calculate the chromatic numbers for A, B, C, D, G, and H. For each, give an example of a vertex coloring of the corresponding graph using exactly χ colors.

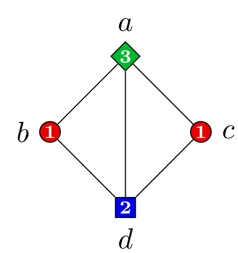
Answer. All of these graphs except for G has a K_3 in it, so must have chromatic number ≥ 3. G has an odd cycle, so cannot be bipartite, i.e. $\chi(G) > 2$. H has a subgraph isomorphic to K_4 (induced by $\{b, c, h, i\}$), and so must have chromatic number ≥ 4. The same upper bounds are verified by the valid colorings given below.

A : 

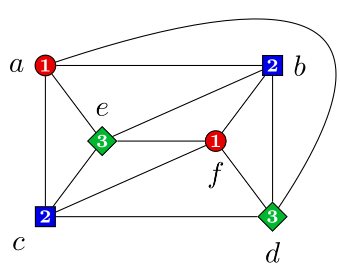
$\chi = 3$

$B =$ 

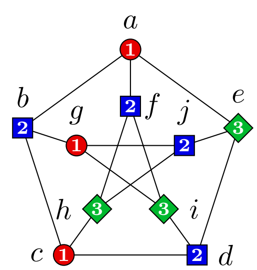
$\chi = 3$

$C =$ 

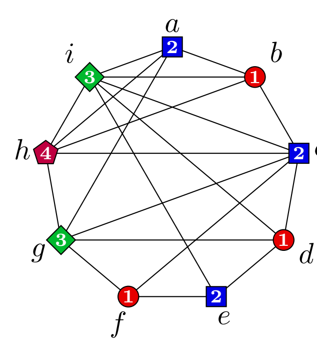
$\chi = 3$

$D =$ 

$\chi = 3$

$G =$ 

$\chi = 3$

$H =$ 

$\chi = 4$

. . . .

(b) Which of A, B, C, D, G, and H have the property that removing a single vertex will reduce the chromatic number?

Answer. None of them have a vertex that would totally disconnect the graph (reduce it to isolated vertices). So none of them can have their chromatic number reduced to 1. However, removing b from A, g from B, and a from C will remove all triangles and drop the chromatic number to 2. Likewise, removing h from H will break the K_4 and drop the chromatic number to 3 (as is evidenced by restricting the given coloring). Finally, D has two disjoint 3-cycles, so removing any one vertex will not drop the chromatic number; similarly for G, which has two disjoint 5-cycles (so that $G - v$ for any vertex still is not bipartite).

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(c) Classify all graphs with chromatic number (i) 1, and (ii) 2.

Answer. If the chromatic number of a graph is 1, then that graph cannot have any edges. So the graphs with chromatic number 1 are those that are a collection of isolated vertices.

If the chromatic number is 2, then the graph is bipartite. So a graph has chromatic number 2 if and only if it has at least one edge, and has no odd cycles.

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(d) What are the chromatic numbers of

(i) K_n

Answer. We have $\chi(K_n) = n$ since every vertex in incident to every other vertex.

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(ii) $K_{m,n}$

Answer. We have $\chi(K_{m,n}) = 2$ since it is bipartite. (Unless m or n is 0, in which case the chromatic number is 1.)

. . . .

(iii) C_n *Answer:* $\chi(C_n) = 2$ if n is even, and 3 if n is odd.

(iv) W_n

Answer. We have $\chi(W_n) = 3$ if n is even (since the cycle is 2-colorable, but the middle vertex is adjacent to everything), and 4 if n is odd (similarly since the cycle is 3-colorable).

. . . .

(v) Q_n *Answer:* $\chi(Q_n) = 2$ since it is bipartite.

Exercise 56. (a) What are the clique and independence numbers of A, B, C, D, G, and H from the previous problem? How do ω and $|V|/\alpha$ compare to χ for each graph?

Answer.

$$\begin{array}{llll}
\omega(A) = 3, & \alpha(A) = 2, & \chi(A) = 3: & \chi = \omega > |V|/\alpha \\
\omega(B) = 3, & \alpha(B) = 3, & \chi(B) = 3: & \chi = \omega > |V|/\alpha \\
\omega(C) = 3, & \alpha(C) = 2, & \chi(C) = 3: & \chi = \omega > |V|/\alpha \\
\omega(D) = 3, & \alpha(D) = 4, & \chi(D) = 3: & \chi = \omega > |V|/\alpha \\
\omega(G) = 2, & \alpha(G) = 4, & \chi(G) = 3: & \chi > \omega, |V|/\alpha \\
\omega(H) = 4, & \alpha(H) = 3, & \chi(H) = 4: & \chi = \omega > |V|/\alpha
\end{array}$$

. . . .

(b) What are the clique and independence numbers of

(i) K_n, (ii) $K_{m,n}$, (iii) C_n, (iv) W_n, (v) Q_n?

How do ω and $|V|/\alpha$ compare to χ for each graph? (You may need to break into cases.)

Answer.

(i) K_n: $\omega = n$, $\alpha = 1$

(ii) $K_{m,n}$: $\omega = 2$, $\alpha = \max(m, n)$.

(iii) C_n: $\omega = 2$, $\alpha = \lfloor n/2 \rfloor$.

(iv) W_n: $\omega = 3$, $\alpha = \lfloor n/2 \rfloor$ (except when $n = 3$, in which case $\omega = 4$, $\alpha = 1$)

(v) Q_n: $\omega = 2$, $\alpha = 2^{n-1}$.

. . . .

(c) Explain why the clique number of the complement of a bipartite is no smaller than the number of vertices in each part. (Recall that the *parts* of a bipartite graph are the two collections of pairwise non-adjacent vertices.)

Answer. The vertices in either part form an independent set. So those vertices in the complement form a clique.

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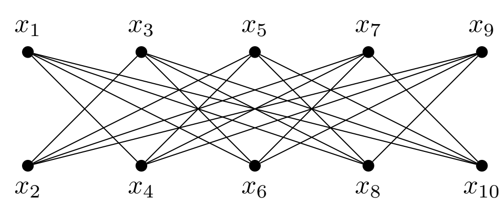
(d) Notice that the graph

$G =$ 

is bipartite, so should have chromatic number 2. Now color this graph using the so-called "greedy algorithm": name your colors "color 1, color 2, . . . ". First color x_1 with color 1; then color x_2 with the lowest color possible (i.e. color 1 if you can, but color 2 if you can't); then color x_3 with the lowest color possible; and so on. How many colors did you need? What is a better way to color G?

You should have used 5 colors.

Exercise 57. (a) The chromatic polynomial for the cycle C_n is $\chi(C_n, t) = (t-1)^n + (-1)^n(t-1)$.

(i) Draw all the ways of coloring the 3-cycle with 3 colors. Then compute $\chi(C_3, 3)$ and compare your answers.

Answer. You can't color C_3 with 2 or fewer colors, so you should get exactly one coloring for each permutation of 3*.* This agrees with

$$\chi(C_3, 3) = (3-1)^3 + (-1)^3(3-1) = 6.$$

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(ii) How many ways are there to color the 5-cycle with 3 colors?

Answer. We have $\chi(C_5, 3) = (3-1)^5 + (-1)^5(3-1) = 30$.

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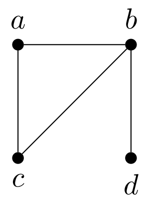
(iii) How many ways are there to color the 6-cycle with 2 colors?

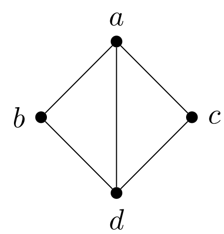
Answer. We have $\chi(C_6, 2) = (2-1)^6 + (-1)^6(2-1) = 2.$

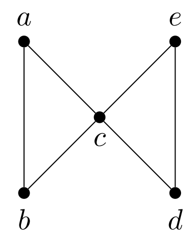
(iv) Use $\chi(C_n, t)$ to verify that even cycles are bipartite and odd cycles are not.

Answer. When n is odd, $\chi(C_n, 2) = -1 + 1 = 0$; when n is even, $\chi(C_n, 2) = 1 + 1 \neq 0$.

(b) For G and H below, compute the number of ways to color the graph with a palate of 1 color, of 2 colors, of 3 colors, and of 4 colors. For K, compute the number of ways to color the graph with a palate of 1 color, of 2 colors, of 3 colors, of 4 colors, and of 5 colors.

$G =$ 

$H =$ 

$K =$ 

Answer. Number of colorings:

t	G	H	K
0	0	0	0
1	0	0	0
2	0	0	0
3	12	6	12
4	72	48	144
5	–	–	720

(c) Calculate the chromatic polynomial for G and for H.

Answer. The facts that both G and H have 0 colorings with palates of 0, 1, and 2-colors gives that $t(t-1)(t-2)$ divides both chromatic polynomials. The fact that both have 4 vertices gives that both chromatic polynomials are of degree 4. Finally, putting these together with the fact that any chromatic polynomial has leading coefficient 1 gives that both chromatic polynomials are of the form

$$t(t-1)^2(t-a) \qquad \text{for some } a.$$

Plugging in $t = 3$, setting equal to 12 and 6, respectively, and solving for a, gives Chromatic polynomials:

$$\chi(G, t) = t(t-1)^2(t-2) = t^4 - 4t^3 + 5t^2 - 2t,$$
$$\chi(H, t) = t(t-1)(t-2)^2 = t^4 - 5t^3 + 8t^2 - 4t.$$

(d) Explain why $\chi(K_n, t) = t(t-1)(t-2)\cdots(t-(n-1))$.

Answer. Given any palate, the possible colors of any vertex depends on every other vertex already colored (since every vertex is adjacent to every other vertex). So there are $P(t, n) = t(t-1)(t-2)\cdots(t-(n-1))$ possible colorings.
