Exercise 55. Let $A$, $B$, $C$, $D$, $G$, and $H$ be the graphs

(a) Calculate the chromatic numbers for $A$, $B$, $C$, $D$, $G$, and $H$. For each, give an example of a vertex coloring of the corresponding graph using exactly $\chi$ colors.

Answer. All of these graphs except for $G$ has a $K_3$ in it, so must have chromatic number $\geq 3$. $G$ has an odd cycle, so cannot be bipartite, i.e. $\chi(G) > 2$. $H$ has a subgraph isomorphic to $K_4$ (induced by $\{b,c,h,i\}$), and so must have chromatic number $\geq 4$. The same upper bounds are verified by the valid colorings given below.
(b) Which of \(A, B, C, D, G\), and \(H\) have the property that removing a single vertex will reduce the chromatic number?

*Answer.* None of them have a vertex that would totally disconnect the graph (reduce it to isolated vertices). So none of them can have their chromatic number reduced to 1. However, removing \(b\) from \(A\), \(g\) from \(B\), and \(a\) from \(C\) will remove all triangles and drop the chromatic number to 2. Likewise, removing \(h\) from \(H\) will break the \(K_4\) and drop the chromatic number to 3 (as is evidenced by restricting the given coloring). Finally, \(D\) has two disjoint 3-cycles, so removing any one vertex will not drop the chromatic number; similarly for \(G\), which has two disjoint 5-cycles (so that \(G - v\) for any vertex still is not bipartite).

(c) Classify all graphs with chromatic number (i) 1, and (ii) 2.

*Answer.* If the chromatic number of a graph is 1, then that graph cannot have any edges. So the graphs with chromatic number 1 are those that are a collection of isolated vertices.

If the chromatic number is 2, then the graph is bipartite. So a graph has chromatic number 2 if and only if it has at least one edge, and has no odd cycles.

(d) What are the chromatic numbers of

(i) \(K_n\)

*Answer.* We have \(\chi(K_n) = n\) since every vertex in incident to every other vertex.

(ii) \(K_{m,n}\)

*Answer.* We have \(\chi(K_{m,n}) = 2\) since it is bipartite. (Unless \(m\) or \(n\) is 0, in which case the chromatic number is 1.)

(iii) \(C_n\)

*Answer:* \(\chi(C_n) = 2\) if \(n\) is even, and 3 if \(n\) is odd.

(iv) \(W_n\)

*Answer.* We have \(\chi(W_n) = 3\) if \(n\) is even (since the cycle is 2-colorable, but the middle vertex is adjacent to everything), and 4 if \(n\) is odd (similarly since the cycle is 3-colorable).

(v) \(Q_n\)

*Answer:* \(\chi(Q_n) = 2\) since it is bipartite.

**Exercise 56.** (a) What are the clique and independence numbers of \(A, B, C, D, G,\) and \(H\) from the previous problem? How do \(\omega\) and \(|V|/\alpha\) compare to \(\chi\) for each graph?

*Answer.*

\[
\begin{align*}
\omega(A) &= 3, & \alpha(A) &= 2, & \chi(A) &= 3 : & \chi = \omega > |V|/\alpha \\
\omega(B) &= 3, & \alpha(B) &= 3, & \chi(B) &= 3 : & \chi = \omega > |V|/\alpha \\
\omega(C) &= 3, & \alpha(C) &= 2, & \chi(C) &= 3 : & \chi = \omega > |V|/\alpha \\
\omega(D) &= 3, & \alpha(D) &= 4, & \chi(D) &= 3 : & \chi = \omega > |V|/\alpha \\
\omega(G) &= 2, & \alpha(G) &= 4, & \chi(G) &= 3 : & \chi > \omega, |V|/\alpha \\
\omega(H) &= 4, & \alpha(H) &= 3, & \chi(H) &= 4 : & \chi = \omega > |V|/\alpha
\end{align*}
\]
(b) What are the clique and independence numbers of
(i) $K_n$, (ii) $K_{m,n}$, (iii) $C_n$, (iv) $W_n$, (v) $Q_n$?

How do $\omega$ and $|V|/\alpha$ compare to $\chi$ for each graph? (You may need to break into cases.)

**Answer.**
(i) $K_n$: $\omega = n$, $\alpha = 1$
(ii) $K_{m,n}$: $\omega = 2$, $\alpha = \max(m, n)$.
(iii) $C_n$: $\omega = 2$, $\alpha = \lfloor n/2 \rfloor$.
(iv) $W_n$: $\omega = 3$, $\alpha = \lfloor n/2 \rfloor$ (except when $n = 3$, in which case $\omega = 4$, $\alpha = 1$)
(v) $Q_n$: $\omega = 2$, $\alpha = 2^{n-1}$.

(c) Explain why the clique number of the complement of a bipartite is no smaller than the number of vertices in each part. (Recall that the parts of a bipartite graph are the two collections of pairwise non-adjacent vertices.)

**Answer.** The vertices in either part form an independent set. So those vertices in the complement form a clique.

(d) Notice that the graph

```
G =
```

is bipartite, so should have chromatic number 2. Now color this graph using the so-called “greedy algorithm”: name your colors “color 1, color 2, . . .”. First color $x_1$ with color 1; then color $x_2$ with the lowest color possible (i.e. color 1 if you can, but color 2 if you can’t); then color $x_3$ with the lowest color possible; and so on. How many colors did you need? What is a better way to color $G$?

You should have used 5 colors.

**Exercise 57.** (a) The chromatic polynomial for the cycle $C_n$ is $\chi(C_n,t) = (t-1)^n + (-1)^n(t-1)$.

(i) Draw all the ways of coloring the 3-cycle with 3 colors. Then compute $\chi(C_3, 3)$ and compare your answers.

**Answer.** You can’t color $C_3$ with 2 or fewer colors, so you should get exactly one coloring for each permutation of 3. This agrees with

$\chi(C_3, 3) = (3 - 1)^3 + (-1)^3(3 - 1) = 6$.

(ii) How many ways are there to color the 5-cycle with 3 colors?

**Answer.** We have $\chi(C_5, 3) = (3 - 1)^5 + (-1)^5(3 - 1) = 30$. 
(iii) How many ways are there to color the 6-cycle with 2 colors?

*Answer.* We have \( \chi(C_6, 2) = (2 - 1)^6 + (-1)^6(2 - 1) = 2. \)

(iv) Use \( \chi(C_n, t) \) to verify that even cycles are bipartite and odd cycles are not.

*Answer.* When \( n \) is odd, \( \chi(C_n, 2) = -1 + 1 = 0; \) when \( n \) is even, \( \chi(C_n, 2) = 1 + 1 \neq 0. \)

(b) For \( G \) and \( H \) below, compute the number of ways to color the graph with a palate of 1 color, of 2 colors, of 3 colors, and of 4 colors. For \( K \), compute the number of ways to color the graph with a palate of 1 color, of 2 colors, of 3 colors, of 4 colors, and of 5 colors.

\[
G = \begin{array}{ccc}
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\end{array}
\]

\[
H = \begin{array}{ccc}
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\end{array}
\]

\[
K = \begin{array}{ccc}
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\end{array}
\]

*Answer.* Number of colorings:

<table>
<thead>
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<th>( t )</th>
<th>( G )</th>
<th>( H )</th>
<th>( K )</th>
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<td>0</td>
<td>0</td>
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</tr>
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</tr>
<tr>
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</tbody>
</table>

(c) Calculate the chromatic polynomial for \( G \) and for \( H \).

*Answer.* The facts that both \( G \) and \( H \) have 0 colorings with palates of 0, 1, and 2-colors gives that \( t(t-1)(t-2) \) divides both chromatic polynomials. The fact that both have 4 vertices gives that both chromatic polynomials are of degree 4. Finally, putting these together with the fact that any chromatic polynomial has leading coefficient 1 gives that both chromatic polynomials are of the form

\[ t(t-1)^2(t-a) \quad \text{for some } a. \]

Plugging in \( t = 3 \), setting equal to 12 and 6, respectively, and solving for \( a \), gives Chromatic polynomials:

\[
\chi(G, t) = t(t-1)^2(t-2) = t^4 - 4t^3 + 5t^2 - 2t;
\]

\[
\chi(H, t) = t(t-1)(t-2)^2 = t^4 - 5t^3 + 8t^2 - 4t.
\]

(d) Explain why \( \chi(K_n, t) = t(t-1)(t-2) \cdots (t-(n-1)). \)

*Answer.* Given any palate, the possible colors of any vertex depends on every other vertex already colored (since every vertex is adjacent to every other vertex). So there are \( P(t, n) = t(t-1)(t-2) \cdots (t-(n-1)) \) possible colorings.