Ch6 supplementary exercises

3. A test contains 100 true/false questions. How many different ways can a student answer the questions on the test, if answers may be left blank?

4. How many strings of length 10 either start with 000 or end with 1111?

5. How many bit strings of length 10 over the alphabet $\{a, b, c\}$ have either exactly three as or exactly four bs?

6. The internal telephone numbers in the phone system on a campus consist of five digits, with the first digit not equal to zero. How many different numbers can be assigned in this system?

7. An ice cream parlor has 28 different flavors, 8 different kinds of sauce, and 12 toppings.

a) In how many different ways can a dish of three scoops of ice cream be made where each flavor can be used more than once and the order of the scoops does not matter?

b) How many different kinds of small sundaes are there if a small sundae contains one scoop of ice cream, a sauce, and a topping?

c) How many different kinds of large sundaes are there if a large sundae contains three scoops of ice cream, where each flavor can be used more than once and the order of the scoops does not matter; two kinds of sauce, where each sauce can be used only once and the order of the sauces does not matter; and three toppings, where each topping can be used only once and the order of the toppings does not matter?

8. How many positive integers less than 1000

a) have exactly three decimal digits? *

b) have an odd number of decimal digits?

c) have at least one decimal digit equal to 9?

d) have no odd decimal digits?

e) have two consecutive decimal digits equal to 5?

f) are palindromes (that is, read the same forward and backward)?

9. When the numbers from 1 to 1000 are written out in decimal notation, how many of each of these digits are used?

a) 0 **b)** 1 **c)** 2 **d)** 9

10. There are 12 signs of the zodiac. How many people are needed to guarantee that at least six of these people have the same sign?

11. A fortune cookie company makes 213 different fortunes. A student eats at a restaurant that uses fortunes from this company and gives each customer one fortune cookie at the end of a meal. What is the largest possible number of times that the student can eat at the restaurant without getting the same fortune four times?

12. How many people are needed to guarantee that at least two were born on the same day of the week and in the same month (perhaps in different years)?

13. Show that given any set of 10 positive integers not exceeding 50 there exist at least two different five-element subsets of this set that have the same sum.

14. A package of baseball cards contains 20 cards. How many packages must be purchased to ensure that two cards in these packages are identical if there are a total of 550 different cards?

15. a) How many cards must be chosen from a standard deck of 52 cards to guarantee that at least two of the four aces are chosen?

b) How many cards must be chosen from a standard deck of 52 cards to guarantee that at least two of the four aces and at least two of the 13 kinds are chosen?

c) How many cards must be chosen from a standard deck of 52 cards to guarantee that there are at least two cards of the same kind?

d) How many cards must be chosen from a standard deck of 52 cards to guarantee that there are at least two cards of each of two different kinds?

* Decimal digit means we write numbers without any leading 0's, i.e. 10=010=0010=… has 2 decimal digits.

20. Once a computer worm infects a personal computer via an infected e-mail message, it sends a copy of itself to 100 e-mail addresses it finds in the electronic message mailbox on this personal computer. What is the maximum number of different computers this one computer can infect in the time it takes for the infected message to be forwarded five times?

21. How many ways are there to choose a dozen donuts from 20 varieties

a) if there are no two donuts of the same variety?

b) if all donuts are of the same variety?

c) if there are no restrictions?

d) if there are at least two varieties among the dozen donuts chosen?

e) if there must be at least six blueberry-filled donuts?

f) if there can be no more than six blueberry-filled donuts?

22. Find n if

a) $P(n, 2) = 110$. **b)** $P(n, n) = 5040$.

c) $P(n, 4) = 12P(n, 2)$.

23. Find n if

a) $C(n, 2) = 45$. **b)** $C(n, 3) = P(n, 2)$.

c) $C(n, 5) = C(n, 2)$.

24. Show that if n and r are nonnegative integers and $n \geq r$, then

$$P(n+1, r) = P(n, r)(n+1)/(n+1-r).$$

***25.** Suppose that S is a set with n elements. How many ordered pairs (A, B) are there such that A and B are subsets of S with $A \subseteq B$? [*Hint:* Show that each element of S belongs to A, $B - A$, or $S - B$.]

26. Give a combinatorial proof of Corollary 2 of Section 6.4 by setting up a correspondence between the subsets of a set with an even number of elements and the subsets of this set with an odd number of elements. [*Hint:* Take an element a in the set. Set up the correspondence by putting a in the subset if it is not already in it and taking it out if it is in the subset.]

27. Let n and r be integers with $1 \leq r < n$. Show that

$$C(n, r-1) = C(n+2, r+1) - 2C(n+1, r+1) + C(n, r+1).$$

28. Prove using mathematical induction that $\sum_{j=2}^{n} C(j, 2) = C(n+1, 3)$ whenever n is an integer greater than 1.

29. Show that if n is an integer then

$$\sum_{k=0}^{n} 3^k \binom{n}{k} = 4^n.$$

***33.** How many bit strings of length n, where $n \geq 4$, contain exactly two occurrences of 01?

35. A professor writes 20 multiple-choice questions, each with the possible answer a, b, c, or d, for a discrete mathematics test. If the number of questions with a, b, c, and d as their answer is 8, 3, 4, and 5, respectively, how many different answer keys are possible, if the questions can be placed in any order?

36. How many different arrangements are there of eight people seated at a round table, where two arrangements are considered the same if one can be obtained from the other by a rotation?

37. How many ways are there to assign 24 students to five faculty advisors?

38. How many ways are there to choose a dozen apples from a bushel containing 20 indistinguishable Delicious apples, 20 indistinguishable Macintosh apples, and 20 indistinguishable Granny Smith apples, if at least three of each kind must be chosen?

39. How many solutions are there to the equation $x_1 + x_2 + x_3 = 17$, where x_1, x_2, and x_3 are nonnegative integers with

a) $x_1 > 1$, $x_2 > 2$, and $x_3 > 3$?
b) $x_1 < 6$ and $x_3 > 5$?
c) $x_1 < 4$, $x_2 < 3$, and $x_3 > 5$?

40. a) How many different strings can be made from the word *PEPPERCORN* when all the letters are used?
b) How many of these strings start and end with the letter P?
c) In how many of these strings are the three letter Ps consecutive?

41. How many subsets of a set with ten elements

a) have fewer than five elements?
b) have more than seven elements?
c) have an odd number of elements?

42. A witness to a hit-and-run accident tells the police that the license plate of the car in the accident, which contains three letters followed by three digits, starts with the letters *AS* and contains both the digits 1 and 2. How many different license plates can fit this description?

43. How many ways are there to put n identical objects into m distinct containers so that no container is empty?

44. How many ways are there to seat six boys and eight girls in a row of chairs so that no two boys are seated next to each other?

45. How many ways are there to distribute six objects to five boxes if

a) both the objects and boxes are labeled?
b) the objects are labeled, but the boxes are unlabeled?
c) the objects are unlabeled, but the boxes are labeled?
d) both the objects and the boxes are unlabeled?

46. How many ways are there to distribute five objects into six boxes if

a) both the objects and boxes are labeled?
b) the objects are labeled, but the boxes are unlabeled?
c) the objects are unlabeled, but the boxes are labeled?
d) both the objects and the boxes are unlabeled?

The **signless Stirling number of the first kind** $c(n, k)$, where k and n are integers with $1 \leq k \leq n$, equals the number of ways to arrange n people around k circular tables with at least one person seated at each table, where two seatings of m people around a circular table are considered the same if everyone has the same left neighbor and the same right neighbor.

47. Find these signless Stirling numbers of the first kind.

a) c(3,2) **b)** c(4,2)
c) c(4,3) **d)** c(5,4)

48. Show that if n is a positive integer, then $\sum_{j=1}^{n} c(n, j) = n!$.

49. Show that if n is a positive integer with $n \geq 3$, then $c(n, n-2) = (3n-1)C(n, 3)/4$.

***50.** Show that if n and k are integers with $1 \leq k < n$, then $c(n+1, k) = c(n, k-1) + nc(n, k)$.

Ch 8
Supplementary Exercises

1. A group of 10 people begin a chain letter, with each person sending the letter to four other people. Each of these people sends the letter to four additional people.
   a) Find a recurrence relation for the number of letters sent at the nth stage of this chain letter, if no person ever receives more than one letter.
   b) What are the initial conditions for the recurrence relation in part (a)?
   c) How many letters are sent at the nth stage of the chain letter?
2. A nuclear reactor has created 18 grams of a particular radioactive isotope. Every hour 1% of this radioactive isotope decays.
   a) Set up a recurrence relation for the amount of this isotope left n hours after its creation.
   b) What are the initial conditions for the recurrence relation in part (a)?
   c) Solve this recurrence relation.
3. Every hour the U.S. government prints 10,000 more \$1 bills, 4000 more \$5 bills, 3000 more \$10 bills, 2500 more \$20 bills, 1000 more \$50 bills, and the same number of \$100 bills as it did the previous hour. In the initial hour 1000 of each bill were produced.
   a) Set up a recurrence relation for the amount of money produced in the nth hour.
   b) What are the initial conditions for the recurrence relation in part (a)?
   c) Solve the recurrence relation for the amount of money produced in the nth hour.
   d) Set up a recurrence relation for the total amount of money produced in the first n hours.
   e) Solve the recurrence relation for the total amount of money produced in the first n hours.
4. Suppose that every hour there are two new bacteria in a colony for each bacterium that was present the previous hour, and that all bacteria 2 hours old die. The colony starts with 100 new bacteria.
   a) Set up a recurrence relation for the number of bacteria present after n hours.
   b) What is the solution of this recurrence relation?
   c) When will the colony contain more than 1 million bacteria?
5. Messages are sent over a communications channel using two different signals. One signal requires 2 microseconds for transmittal, and the other signal requires 3 microseconds for transmittal. Each signal of a message is followed immediately by the next signal.
   a) Find a recurrence relation for the number of different signals that can be sent in n microseconds.
   b) What are the initial conditions of the recurrence relation in part (a)?
   c) How many different messages can be sent in 12 microseconds?
6. A small post office has only 4-cent stamps, 6-cent stamps, and 10-cent stamps. Find a recurrence relation for the number of ways to form postage of n cents with these stamps if the order that the stamps are used matters. What are the initial conditions for this recurrence relation?
7. How many ways are there to form these postages using the rules described in Exercise 6?
   a) 12 cents
   b) 14 cents
   c) 18 cents
   d) 22 cents

*

11. Find the solution of the recurrence relation $a_n = 3a_{n-1} - 3a_{n-2} + a_{n-3} + 1$ if $a_0 = 2$, $a_1 = 4$, and $a_2 = 8$.
12. Find the solution of the recurrence relation $a_n = 3a_{n-1} - 3a_{n-2} + a_{n-3}$ if $a_0 = 2$, $a_1 = 2$, and $a_2 = 4$.

*13. Suppose that in Example 1 of Section 8.1 a pair of rabbits leaves the island after reproducing twice. Find a recurrence relation for the number of rabbits on the island in the middle of the nth month.

8.2 Exercises

1. Determine which of these are linear homogeneous recurrence relations with constant coefficients. Also, find the degree of those that are.

a) $a_n = 3a_{n-1} + 4a_{n-2} + 5a_{n-3}$
b) $a_n = 2na_{n-1} + a_{n-2}$
c) $a_n = a_{n-1} + a_{n-4}$
d) $a_n = a_{n-1} + 2$
e) $a_n = a_{n-1}^2 + a_{n-2}$
f) $a_n = a_{n-2}$
g) $a_n = a_{n-1} + n$

2. Determine which of these are linear homogeneous recurrence relations with constant coefficients. Also, find the degree of those that are.

a) $a_n = 3a_{n-2}$
b) $a_n = 3$
c) $a_n = a_{n-1}^2$
d) $a_n = a_{n-1} + 2a_{n-3}$
e) $a_n = a_{n-1}/n$
f) $a_n = a_{n-1} + a_{n-2} + n + 3$
g) $a_n = 4a_{n-2} + 5a_{n-4} + 9a_{n-7}$

3. Solve these recurrence relations together with the initial conditions given.

a) $a_n = 2a_{n-1}$ for $n \geq 1$, $a_0 = 3$
b) $a_n = a_{n-1}$ for $n \geq 1$, $a_0 = 2$
c) $a_n = 5a_{n-1} - 6a_{n-2}$ for $n \geq 2$, $a_0 = 1$, $a_1 = 0$
d) $a_n = 4a_{n-1} - 4a_{n-2}$ for $n \geq 2$, $a_0 = 6$, $a_1 = 8$
e) $a_n = -4a_{n-1} - 4a_{n-2}$ for $n \geq 2$, $a_0 = 0$, $a_1 = 1$
f) $a_n = 4a_{n-2}$ for $n \geq 2$, $a_0 = 0$, $a_1 = 4$
g) $a_n = a_{n-2}/4$ for $n \geq 2$, $a_0 = 1$, $a_1 = 0$

4. Solve these recurrence relations together with the initial conditions given.

a) $a_n = a_{n-1} + 6a_{n-2}$ for $n \geq 2$, $a_0 = 3$, $a_1 = 6$
b) $a_n = 7a_{n-1} - 10a_{n-2}$ for $n \geq 2$, $a_0 = 2$, $a_1 = 1$
c) $a_n = 6a_{n-1} - 8a_{n-2}$ for $n \geq 2$, $a_0 = 4$, $a_1 = 10$
d) $a_n = 2a_{n-1} - a_{n-2}$ for $n \geq 2$, $a_0 = 4$, $a_1 = 1$
e) $a_n = a_{n-2}$ for $n \geq 2$, $a_0 = 5$, $a_1 = -1$
f) $a_n = -6a_{n-1} - 9a_{n-2}$ for $n \geq 2$, $a_0 = 3$, $a_1 = -3$
g) $a_{n+2} = -4a_{n+1} + 5a_n$ for $n \geq 0$, $a_0 = 2$, $a_1 = 8$

5. How many different messages can be transmitted in n microseconds using the two signals described in Exercise 19 in Section 8.1?

6. How many different messages can be transmitted in n microseconds using three different signals if one signal requires 1 microsecond for transmittal, the other two signals require 2 microseconds each for transmittal, and a signal in a message is followed immediately by the next signal?

7. In how many ways can a $2 \times n$ rectangular checkerboard be tiled using 1×2 and 2×2 pieces?

8. A model for the number of lobsters caught per year is based on the assumption that the number of lobsters caught in a year is the average of the number caught in the two previous years.

a) Find a recurrence relation for $\{L_n\}$, where L_n is the number of lobsters caught in year n, under the assumption for this model.

b) Find L_n if 100,000 lobsters were caught in year 1 and 300,000 were caught in year 2.

9. A deposit of \$100,000 is made to an investment fund at the beginning of a year. On the last day of each year two dividends are awarded. The first dividend is 20% of the amount in the account during that year. The second dividend is 45% of the amount in the account in the previous year.

a) Find a recurrence relation for $\{P_n\}$, where P_n is the amount in the account at the end of n years if no money is ever withdrawn.

b) How much is in the account after n years if no money has been withdrawn?

***10.** Prove Theorem 2.

11. The **Lucas numbers** satisfy the recurrence relation

$$L_n = L_{n-1} + L_{n-2},$$

and the initial conditions $L_0 = 2$ and $L_1 = 1$.

a) Show that $L_n = f_{n-1} + f_{n+1}$ for $n = 2, 3, \ldots,$ where f_n is the nth Fibonacci number.

b) Find an explicit formula for the Lucas numbers.

12. Find the solution to $a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$ for $n = 3, 4, 5, \ldots,$ with $a_0 = 3$, $a_1 = 6$, and $a_2 = 0$.

13. Find the solution to $a_n = 7a_{n-2} + 6a_{n-3}$ with $a_0 = 9$, $a_1 = 10$, and $a_2 = 32$.

14. Find the solution to $a_n = 5a_{n-2} - 4a_{n-4}$ with $a_0 = 3$, $a_1 = 2$, $a_2 = 6$, and $a_3 = 8$.

15. Find the solution to $a_n = 2a_{n-1} + 5a_{n-2} - 6a_{n-3}$ with $a_0 = 7$, $a_1 = -4$, and $a_2 = 8$.

***16.** Prove Theorem 3.

17. Prove this identity relating the Fibonacci numbers and the binomial coefficients:

$$f_{n+1} = C(n, 0) + C(n-1, 1) + \cdots + C(n-k, k),$$

where n is a positive integer and $k = \lfloor n/2 \rfloor$. [*Hint:* Let $a_n = C(n, 0) + C(n-1, 1) + \cdots + C(n-k, k)$. Show that the sequence $\{a_n\}$ satisfies the same recurrence relation and initial conditions satisfied by the sequence of Fibonacci numbers.]

18. Solve the recurrence relation $a_n = 6a_{n-1} - 12a_{n-2} + 8a_{n-3}$ with $a_0 = -5$, $a_1 = 4$, and $a_2 = 88$.

19. Solve the recurrence relation $a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$ with $a_0 = 5$, $a_1 = -9$, and $a_2 = 15$.

20. Find the general form of the solutions of the recurrence relation $a_n = 8a_{n-2} - 16a_{n-4}$.

21. What is the general form of the solutions of a linear homogeneous recurrence relation if its characteristic equation has roots $1, 1, 1, 1, -2, -2, -2, 3, 3, -4$?

22. What is the general form of the solutions of a linear homogeneous recurrence relation if its characteristic equation has the roots $-1, -1, -1, 2, 2, 5, 5, 7$?

23. Consider the nonhomogeneous linear recurrence relation $a_n = 3a_{n-1} + 2^n$.

a) Show that $a_n = -2^{n+1}$ is a solution of this recurrence relation.

b) Use Theorem 5 to find all solutions of this recurrence relation.

c) Find the solution with $a_0 = 1$.

24. Consider the nonhomogeneous linear recurrence relation $a_n = 2a_{n-1} + 2^n$.

a) Show that $a_n = n2^n$ is a solution of this recurrence relation.

b) Use Theorem 5 to find all solutions of this recurrence relation.

c) Find the solution with $a_0 = 2$.

25. a) Determine values of the constants A and B such that $a_n = An + B$ is a solution of recurrence relation $a_n = 2a_{n-1} + n + 5$.

b) Use Theorem 5 to find all solutions of this recurrence relation.

c) Find the solution of this recurrence relation with $a_0 = 4$.

26. What is the general form of the particular solution guaranteed to exist by Theorem 6 of the linear nonhomogeneous recurrence relation $a_n = 6a_{n-1} - 12a_{n-2} + 8a_{n-3} + F(n)$ if

a) $F(n) = n^2$? **b)** $F(n) = 2^n$?
c) $F(n) = n2^n$? **d)** $F(n) = (-2)^n$?
e) $F(n) = n^2 2^n$? **f)** $F(n) = n^3(-2)^n$?
g) $F(n) = 3$?

27. What is the general form of the particular solution guaranteed to exist by Theorem 6 of the linear nonhomogeneous recurrence relation $a_n = 8a_{n-2} - 16a_{n-4} + F(n)$ if

a) $F(n) = n^3$? **b)** $F(n) = (-2)^n$?
c) $F(n) = n2^n$? **d)** $F(n) = n^2 4^n$?
e) $F(n) = (n^2 - 2)(-2)^n$? **f)** $F(n) = n^4 2^n$?
g) $F(n) = 2$?

28. a) Find all solutions of the recurrence relation $a_n = 2a_{n-1} + 2n^2$.

b) Find the solution of the recurrence relation in part (a) with initial condition $a_1 = 4$.

29. a) Find all solutions of the recurrence relation $a_n = 2a_{n-1} + 3^n$.

b) Find the solution of the recurrence relation in part (a) with initial condition $a_1 = 5$.

30. a) Find all solutions of the recurrence relation $a_n = -5a_{n-1} - 6a_{n-2} + 42 \cdot 4^n$.

b) Find the solution of this recurrence relation with $a_1 = 56$ and $a_2 = 278$.

31. Find all solutions of the recurrence relation $a_n = 5a_{n-1} - 6a_{n-2} + 2^n + 3n$. [*Hint:* Look for a particular solution of the form $qn2^n + p_1 n + p_2$, where q, p_1, and p_2 are constants.]

32. Find the solution of the recurrence relation $a_n = 2a_{n-1} + 3 \cdot 2^n$.

33. Find all solutions of the recurrence relation $a_n = 4a_{n-1} - 4a_{n-2} + (n+1)2^n$.

8.4 Exercises

1. Find the generating function for the finite sequence 2, 2, 2, 2, 2, 2.
2. Find the generating function for the finite sequence 1, 4, 16, 64, 256.

In Exercises 3–8, by a **closed form** we mean an algebraic expression not involving a summation over a range of values or the use of ellipses.

3. Find a closed form for the generating function for each of these sequences. (For each sequence, use the most obvious choice of a sequence that follows the pattern of the initial terms listed.)
   a) 0, 2, 2, 2, 2, 2, 2, 0, 0, 0, 0, 0, . . .
   b) 0, 0, 0, 1, 1, 1, 1, 1, 1, . . .
   c) 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, . . .
   d) 2, 4, 8, 16, 32, 64, 128, 256, . . .
   e) $\binom{7}{0}, \binom{7}{1}, \binom{7}{2}, \ldots, \binom{7}{7}$, 0, 0, 0, 0, 0, . . .
   f) 2, −2, 2, −2, 2, −2, 2, −2, . . .
   g) 1, 1, 0, 1, 1, 1, 1, 1, 1, 1, . . .
   h) 0, 0, 0, 1, 2, 3, 4, . . .
4. Find a closed form for the generating function for each of these sequences. (Assume a general form for the terms of the sequence, using the most obvious choice of such a sequence.)
   a) −1, −1, −1, −1, −1, −1, −1, 0, 0, 0, 0, 0, 0, . . .
   b) 1, 3, 9, 27, 81, 243, 729, . . .
   c) 0, 0, 3, −3, 3, −3, 3, −3, . . .
   d) 1, 2, 1, 1, 1, 1, 1, 1, 1, . . .
   e) $\binom{7}{0}, 2\binom{7}{1}, 2^2\binom{7}{2}, \ldots, 2^7\binom{7}{7}$, 0, 0, 0, 0, . . .
   f) −3, 3, −3, 3, −3, 3, . . .
   g) 0, 1, −2, 4, −8, 16, −32, 64, . . .
   h) 1, 0, 1, 0, 1, 0, 1, 0, . . .
5. Find a closed form for the generating function for the sequence $\{a_n\}$, where
   a) $a_n = 5$ for all $n = 0, 1, 2, \ldots$.
   b) $a_n = 3^n$ for all $n = 0, 1, 2, \ldots$.
   c) $a_n = 2$ for $n = 3, 4, 5, \ldots$ and $a_0 = a_1 = a_2 = 0$.
   d) $a_n = 2n + 3$ for all $n = 0, 1, 2, \ldots$.
   e) $a_n = \binom{8}{n}$ for all $n = 0, 1, 2, \ldots$.
   f) $a_n = \binom{n+4}{n}$ for all $n = 0, 1, 2, \ldots$.
6. Find a closed form for the generating function for the sequence $\{a_n\}$, where
   a) $a_n = -1$ for all $n = 0, 1, 2, \ldots$.
   b) $a_n = 2^n$ for $n = 1, 2, 3, 4, \ldots$ and $a_0 = 0$.
   c) $a_n = n - 1$ for $n = 0, 1, 2, \ldots$.
   d) $a_n = 1/(n+1)!$ for $n = 0, 1, 2, \ldots$.
   e) $a_n = \binom{n}{2}$ for $n = 0, 1, 2, \ldots$.
   f) $a_n = \binom{10}{n+1}$ for $n = 0, 1, 2, \ldots$.
7. For each of these generating functions, provide a closed formula for the sequence it determines.
   a) $(3x-4)^3$
   b) $(x^3+1)^3$
   c) $1/(1-5x)$
   d) $x^3/(1+3x)$
   e) $x^2 + 3x + 7 + (1/(1-x^2))$
   f) $(x^4/(1-x^4)) - x^3 - x^2 - x - 1$
   g) $x^2/(1-x)^2$
   h) $2e^{2x}$
8. For each of these generating functions, provide a closed formula for the sequence it determines.
   a) $(x^2+1)^3$
   b) $(3x-1)^3$
   c) $1/(1-2x^2)$
   d) $x^2/(1-x)^3$
   e) $x - 1 + (1/(1-3x))$
   f) $(1+x^3)/(1+x)^3$
   *g) $x/(1+x+x^2)$
   h) $e^{3x^2} - 1$
9. Find the coefficient of x^{10} in the power series of each of these functions.
   a) $(1 + x^5 + x^{10} + x^{15} + \cdots)^3$
   b) $(x^3 + x^4 + x^5 + x^6 + x^7 + \cdots)^3$
   c) $(x^4 + x^5 + x^6)(x^3 + x^4 + x^5 + x^6 + x^7)(1 + x + x^2 + x^3 + x^4 + \cdots)$
   d) $(x^2 + x^4 + x^6 + x^8 + \cdots)(x^3 + x^6 + x^9 + \cdots)(x^4 + x^8 + x^{12} + \cdots)$
   e) $(1 + x^2 + x^4 + x^6 + x^8 + \cdots)(1 + x^4 + x^8 + x^{12} + \cdots)(1 + x^6 + x^{12} + x^{18} + \cdots)$
10. Find the coefficient of x^9 in the power series of each of these functions.
    a) $(1 + x^3 + x^6 + x^9 + \cdots)^3$
    b) $(x^2 + x^3 + x^4 + x^5 + x^6 + \cdots)^3$
    c) $(x^3 + x^5 + x^6)(x^3 + x^4)(x + x^2 + x^3 + x^4 + \cdots)$
    d) $(x + x^4 + x^7 + x^{10} + \cdots)(x^2 + x^4 + x^6 + x^8 + \cdots)$
    e) $(1 + x + x^2)^3$
11. Find the coefficient of x^{10} in the power series of each of these functions.
    a) $1/(1-2x)$
    b) $1/(1+x)^2$
    c) $1/(1-x)^3$
    d) $1/(1+2x)^4$
    e) $x^4/(1-3x)^3$
12. Find the coefficient of x^{12} in the power series of each of these functions.
    a) $1/(1+3x)$
    b) $1/(1-2x)^2$
    c) $1/(1+x)^8$
    d) $1/(1-4x)^3$
    e) $x^3/(1+4x)^2$
13. Use generating functions to determine the number of different ways 10 identical balloons can be given to four children if each child receives at least two balloons.
14. Use generating functions to determine the number of different ways 12 identical action figures can be given to five children so that each child receives at most three action figures.
15. Use generating functions to determine the number of different ways 15 identical stuffed animals can be given to six children so that each child receives at least one but no more than three stuffed animals.

16. Use generating functions to find the number of ways to choose a dozen bagels from three varieties—egg, salty, and plain—if at least two bagels of each kind but no more than three salty bagels are chosen.

17. In how many ways can 25 identical donuts be distributed to four police officers so that each officer gets at least three but no more than seven donuts?

18. Use generating functions to find the number of ways to select 14 balls from a jar containing 100 red balls, 100 blue balls, and 100 green balls so that no fewer than 3 and no more than 10 blue balls are selected. Assume that the order in which the balls are drawn does not matter.

19. What is the generating function for the sequence $\{c_k\}$, where c_k is the number of ways to make change for k dollars using \$1 bills, \$2 bills, \$5 bills, and \$10 bills?

20. What is the generating function for the sequence $\{c_k\}$, where c_k represents the number of ways to make change for k pesos using bills worth 10 pesos, 20 pesos, 50 pesos, and 100 pesos?

21. Give a combinatorial interpretation of the coefficient of x^4 in the expansion $(1 + x + x^2 + x^3 + \cdots)^3$. Use this interpretation to find this number.

22. Give a combinatorial interpretation of the coefficient of x^6 in the expansion $(1 + x + x^2 + x^3 + \cdots)^n$. Use this interpretation to find this number.

23. **a)** What is the generating function for $\{a_k\}$, where a_k is the number of solutions of $x_1 + x_2 + x_3 = k$ when x_1, x_2, and x_3 are integers with $x_1 \geq 2$, $0 \leq x_2 \leq 3$, and $2 \leq x_3 \leq 5$?

b) Use your answer to part (a) to find a_6.

24. **a)** What is the generating function for $\{a_k\}$, where a_k is the number of solutions of $x_1 + x_2 + x_3 + x_4 = k$ when x_1, x_2, x_3, and x_4 are integers with $x_1 \geq 3$, $1 \leq x_2 \leq 5$, $0 \leq x_3 \leq 4$, and $x_4 \geq 1$?

b) Use your answer to part (a) to find a_7.

25. Explain how generating functions can be used to find the number of ways in which postage of r cents can be pasted on an envelope using 3-cent, 4-cent, and 20-cent stamps.

a) Assume that the order the stamps are pasted on does not matter.

b) Assume that the stamps are pasted in a row and the order in which they are pasted on matters.

c) Use your answer to part (a) to determine the number of ways 46 cents of postage can be pasted on an envelope using 3-cent, 4-cent, and 20-cent stamps when the order the stamps are pasted on does not matter. (Use of a computer algebra program is advised.)

d) Use your answer to part (b) to determine the number of ways 46 cents of postage can be pasted in a row on an envelope using 3-cent, 4-cent, and 20-cent stamps when the order in which the stamps are pasted on matters. (Use of a computer algebra program is advised.)

26. **a)** Show that $1/(1 - x - x^2 - x^3 - x^4 - x^5 - x^6)$ is the generating function for the number of ways that the sum n can be obtained when a die is rolled repeatedly and the order of the rolls matters.

b) Use part (a) to find the number of ways to roll a total of 8 when a die is rolled repeatedly, and the order of the rolls matters. (Use of a computer algebra package is advised.)

27. Use generating functions (and a computer algebra package, if available) to find the number of ways to make change for \$1 using

a) dimes and quarters.

b) nickels, dimes, and quarters.

c) pennies, dimes, and quarters.

d) pennies, nickels, dimes, and quarters.

28. Use generating functions (and a computer algebra package, if available) to find the number of ways to make change for \$1 using pennies, nickels, dimes, and quarters with

a) no more than 10 pennies.

b) no more than 10 pennies and no more than 10 nickels.

***c)** no more than 10 coins.

29. Use generating functions to find the number of ways to make change for \$100 using

a) \$10, \$20, and \$50 bills.

b) \$5, \$10, \$20, and \$50 bills.

c) \$5, \$10, \$20, and \$50 bills if at least one bill of each denomination is used.

d) \$5, \$10, and \$20 bills if at least one and no more than four of each denomination is used.

30. If $G(x)$ is the generating function for the sequence $\{a_k\}$, what is the generating function for each of these sequences?

a) $2a_0, 2a_1, 2a_2, 2a_3, \ldots$

b) $0, a_0, a_1, a_2, a_3, \ldots$ (assuming that terms follow the pattern of all but the first term)

c) $0, 0, 0, 0, a_2, a_3, \ldots$ (assuming that terms follow the pattern of all but the first four terms)

d) $a_2, a_3, a_4, \ldots$

e) $a_1, 2a_2, 3a_3, 4a_4, \ldots$ [*Hint: Calculus required* here.]

f) $a_0^2,\ 2a_0a_1,\ a_1^2 + 2a_0a_2,\ 2a_0a_3 + 2a_1a_2,\ 2a_0a_4 + 2a_1a_3 + a_2^2, \ldots$

31. If $G(x)$ is the generating function for the sequence $\{a_k\}$, what is the generating function for each of these sequences?

a) $0, 0, 0, a_3, a_4, a_5, \ldots$ (assuming that terms follow the pattern of all but the first three terms)

b) $a_0, 0, a_1, 0, a_2, 0, \ldots$

c) $0, 0, 0, 0, a_0, a_1, a_2, \ldots$ (assuming that terms follow the pattern of all but the first four terms)

d) $a_0, 2a_1, 4a_2, 8a_3, 16a_4, \ldots$

e) $0, a_0, a_1/2, a_2/3, a_3/4, \ldots$ [*Hint: Calculus required* here.]

f) $a_0, a_0 + a_1, a_0 + a_1 + a_2, a_0 + a_1 + a_2 + a_3, \ldots$

32. Use generating functions to solve the recurrence relation $a_k = 7a_{k-1}$ with the initial condition $a_0 = 5$.

33. Use generating functions to solve the recurrence relation $a_k = 3a_{k-1} + 2$ with the initial condition $a_0 = 1$.

34. Use generating functions to solve the recurrence relation $a_k = 3a_{k-1} + 4^{k-1}$ with the initial condition $a_0 = 1$.

35. Use generating functions to solve the recurrence relation $a_k = 5a_{k-1} - 6a_{k-2}$ with initial conditions $a_0 = 6$ and $a_1 = 30$.

36. Use generating functions to solve the recurrence relation $a_k = a_{k-1} + 2a_{k-2} + 2^k$ with initial conditions $a_0 = 4$ and $a_1 = 12$.

37. Use generating functions to solve the recurrence relation $a_k = 4a_{k-1} - 4a_{k-2} + k^2$ with initial conditions $a_0 = 2$ and $a_1 = 5$.

38. Use generating functions to solve the recurrence relation $a_k = 2a_{k-1} + 3a_{k-2} + 4^k + 6$ with initial conditions $a_0 = 20$, $a_1 = 60$.

39. Use generating functions to find an explicit formula for the Fibonacci numbers.

***40.** **a)** Show that if n is a positive integer, then

$$\binom{-1/2}{n} = \frac{\binom{2n}{n}}{(-4)^n}.$$

b) Use the extended binomial theorem and part (a) to show that the coefficient of x^n in the expansion of $(1-4x)^{-1/2}$ is $\binom{2n}{n}$ for all nonnegative integers n.

***41.** (*Calculus required*) Let $\{C_n\}$ be the sequence of Catalan numbers, that is, the solution to the recurrence relation $C_n = \sum_{k=0}^{n-1} C_k C_{n-k-1}$ with $C_0 = C_1 = 1$ (see Example 5 in Section 8.1).

a) Show that if $G(x)$ is the generating function for the sequence of Catalan numbers, then $xG(x)^2 - G(x) + 1 = 0$. Conclude (using the initial conditions) that $G(x) = (1 - \sqrt{1-4x})/(2x)$.

b) Use Exercise 40 to conclude that

$$G(x) = \sum_{n=0}^{\infty} \frac{1}{n+1} \binom{2n}{n} x^n,$$

so that

$$C_n = \frac{1}{n+1} \binom{2n}{n}.$$

c) Show that $C_n \geq 2^{n-1}$ for all positive integers n.

42. Use generating functions to prove Pascal's identity: $C(n, r) = C(n-1, r) + C(n-1, r-1)$ when n and r are positive integers with $r < n$. [*Hint:* Use the identity $(1+x)^n = (1+x)^{n-1} + x(1+x)^{n-1}$.]

43. Use generating functions to prove Vandermonde's identity: $C(m+n, r) = \sum_{k=0}^{r} C(m, r-k)C(n, k)$, whenever m, n, and r are nonnegative integers with r not exceeding either m or n. [*Hint:* Look at the coefficient of x^r in both sides of $(1+x)^{m+n} = (1+x)^m(1+x)^n$.]

44. This exercise shows how to use generating functions to derive a formula for the sum of the first n squares.

a) Show that $(x^2 + x)/(1-x)^4$ is the generating function for the sequence $\{a_n\}$, where $a_n = 1^2 + 2^2 + \cdots + n^2$.

b) Use part (a) to find an explicit formula for the sum $1^2 + 2^2 + \cdots + n^2$.

The **exponential generating function** for the sequence $\{a_n\}$ is the series

$$\sum_{n=0}^{\infty} \frac{a_n}{n!} x^n.$$

For example, the exponential generating function for the sequence 1, 1, 1, ... is the function $\sum_{n=0}^{\infty} x^n/n! = e^x$. (You will find this particular series useful in these exercises.) Note that e^x is the (ordinary) generating function for the sequence 1, 1, 1/2!, 1/3!, 1/4!, ...

45. Find a closed form for the exponential generating function for the sequence $\{a_n\}$, where

a) $a_n = 2$.
b) $a_n = (-1)^n$.
c) $a_n = 3^n$.
d) $a_n = n + 1$.
e) $a_n = 1/(n+1)$.

46. Find a closed form for the exponential generating function for the sequence $\{a_n\}$, where

a) $a_n = (-2)^n$.
b) $a_n = -1$.
c) $a_n = n$.
d) $a_n = n(n-1)$.
e) $a_n = 1/((n+1)(n+2))$.

Generating functions are useful in studying the number of different types of partitions of an integer n. A **partition** of a positive integer is a way to write this integer as the sum of positive integers where repetition is allowed and the order of the integers in the sum does not matter. For example, the partitions of 5 (with no restrictions) are $1+1+1+1+1$, $1+1+1+2$, $1+1+3$, $1+2+2$, $1+4$, $2+3$, and 5. Exercises 51–56 illustrate some of these uses.

51. Show that the coefficient $p(n)$ of x^n in the formal power series expansion of $1/((1-x)(1-x^2)(1-x^3)\cdots)$ equals the number of partitions of n.

52. Show that the coefficient $p_o(n)$ of x^n in the formal power series expansion of $1/((1-x)(1-x^3)(1-x^5)\cdots)$ equals the number of partitions of n into odd integers, that is, the number of ways to write n as the sum of odd positive integers, where the order does not matter and repetitions are allowed.

53. Show that the coefficient $p_d(n)$ of x^n in the formal power series expansion of $(1+x)(1+x^2)(1+x^3)\cdots$ equals the number of partitions of n into distinct parts, that is, the number of ways to write n as the sum of positive integers, where the order does not matter but no repetitions are allowed.

54. Find $p_o(n)$, the number of partitions of n into odd parts with repetitions allowed, and $p_d(n)$, the number of partitions of n into distinct parts, for $1 \le n \le 8$, by writing each partition of each type for each integer.

55. Show that if n is a positive integer, then the number of partitions of n into distinct parts equals the number of partitions of n into odd parts with repetitions allowed; that is, $p_o(n) = p_d(n)$. [*Hint:* Show that the generating functions for $p_o(n)$ and $p_d(n)$ are equal.]