Answers (not solutions) for Ch 6 Supp. Exercises (pp 441-443)

3) $3^{10}$

4) $2^7 + 2^6 - 2^3$

5) $(\frac{10}{3})2^7 + (\frac{10}{4})2^6 - (\frac{10}{3})(\frac{10-3}{4})10^4 - 3$

6) $9 \cdot 10^4$

7) (a) \( \binom{3+28-1}{3} \)
   (b) $28 \cdot 8 \cdot 12$
   (c) \( \binom{3+28-1}{3}(3)(12) \)

8) (a) $9 \cdot 10^2$
   (b) $9 \cdot 10^2 + 9$
   (c) $1000 - (1 + 9^3 - 1)$
   (d) $5^3 - 1$
   (e) $9 + 9 + 1$
   (f) $9 + 9 \cdot 10$
9) (a) \[ \frac{8}{9} + 9 \cdot 9 + 9 \cdot 10 + 9 \cdot 10^2 + 3 \]
\[
= \frac{1}{9} + \frac{2}{9} + -\frac{3}{9} + \frac{-1}{9} + \frac{-1}{9} + \frac{1}{100}
\]
(b) \[ 1 + 9 + 10 + 9 \cdot 10 + 9 \cdot 10^2 + 1 \]
\[ 1 + \frac{9}{10} + \frac{-1}{10} + \frac{-1}{10} + \frac{1}{100} \]

(c) (d) \[ 1 + 9 + 10 + 9 \cdot 10 + 9 \cdot 10^2 \]

10) Pigeonhole: \[ \lceil \sqrt[2]{N} \rceil > 6 \iff \sqrt[2]{N} > 5 \iff N > 5^2 + 1 \]
\[ N = 5^2 + 1 \]

11) 3.213

12) 7.12+1

13) 10 positive integers \leq 50: A = \{ a_1, a_2, ..., a_{10} \} \text{ where } a_i \leq 50 \}

\# 5-element subsets: \( \binom{10}{5} = 252 \)

Possible sums of elements from \( 9, 10, ..., 50 \) :

\begin{align*}
\text{min:} & \quad 1+2+3+4+5 = 15 \\
\text{max:} & \quad 46+47+48+49+50 = 240 \\
\text{total possibilities:} & \quad 240 - 15 + 1 = 226
\end{align*}

Since \( \# \{ \text{5-elt subsets of } A \} > \# \{ \text{sums of 5-elt subsets of } 9, 10, ..., 50 \} \)

At least two 5-elt subsets of \( A \) must have the same sum.

14) \[ \left\lfloor \frac{20 \cdot k}{550} \right\rfloor > 2 \iff \frac{20 \cdot k}{550} > 1 \iff k > \frac{55}{2} \text{, so } k = 28 \]
15) \(\omega(52-4)+2\)

(b) \(\max\left(\frac{52-4+2}{4+1}\right) = 52-4+2\)

(c) \(13+1\)

(d) \(13+3+1\)

20) \(100^5\)

21) (a) \(\binom{20}{12}\)

(b) \(\binom{20}{1} = 20\)

(c) \(\binom{12+20-1}{12}\)

(d) \(\binom{12+20-1}{12} - 20\)

(e) \(\binom{12-6+20-1}{12-6}\)

(f) \(\sum_{k=0}^{6} \binom{12-k+(20-1)-1}{12-k}\)

↑ \(k = \# \text{ blueberry - filled} \), \(12-k = \# \text{ remaining choices} \)

\(20-1 = \# \text{ varieties not bb-filled}\)
22) (a) \( 110 = P(n, 2) = n(n-1) \Rightarrow n = 11 \)  
(solve \( n^2 - n - 110 = 0 \); \( n = 11, -10 \)  
\( n = 11 \))

(b) \( 5040 = P(n, 3) = n! \Rightarrow n = 7 \)

(c) \( n(n-1)(n-2)(n-3) = P(n, 4) \)
\( = P(n, 2) \cdot 12 \)
\( = 12(n\times n-1) \)
so \( (n-2)(n-3) = 12 \)
\( n^2 - 5n + 6 = 0 \)
\( \frac{-(-5) \pm \sqrt{(-5)^2 - 4 \cdot 1 \cdot 6}}{2} \)
\( (n-6)(n+1) = 0 \)
so \( n = 6 \)  
\( \text{not pos} \)

23) (a) \( 45 = C(n, 2) = \frac{1}{2} \cdot n(n-1) \Rightarrow n = 10 \)

(b) \( \frac{1}{6} n(n-1)(n-2) = C(n, 3) \)
\( = P(n, 2) = n(n-1) \)
so \( \frac{1}{6} (n-2) = 1 \)
\( n-2 = 6 \)
\( n = 8 \)

(c) \( n(n-1)(n-2)(n-3)(n-4) \cdot \frac{1}{5!} \)
\( = C(n, 5) = C(n, 2) = \frac{1}{2} n(n-1) \)

\( \text{Stop! This only happens in } \binom{n}{k} = \binom{n-k}{k} \)  
\( 5 = n-2 \Rightarrow n = 7 \)
24) Show $n, r \in \mathbb{Z}_+ \implies n \geq r$, then

$$P(n+1, r) = \frac{P(n, r)(n+1)}{n+1-r}$$

**Combinatorial Proof Outline:**

LHS: # $r$-permutations of $n+1$ things

RHS: Build an $r$-perm of $\{1, \ldots, n+1\}$ as follows:

1. Pick 1st elt: $n+1$ ways
   (ex: 3)

2. Add on $r$-perm of remaining $(n+1)-1=n$ els:
   $P(n, r)$ ways
   (ex: 32145)

3. Forget $(r+1)^{th}$ elt of the resulting $(r+1)^{th}$-perm of $n+1$ things
   (ex: 32145)

Since there are for any fixed $r$-perm of $n+1$, a total of $n+1-r$ things that forgotten last elt could have been, there are

$$\frac{(n+1)P(n, r)}{n+1-r} \quad r$-$perms$ of $n+1$.

**Algebraic Proof**

$$\frac{P(n, r)(n+1)}{n+1-r} = \frac{P(n, r)}{n+1-r} \cdot \frac{(n+1)n(n-1)\ldots(n+1-r)}{n+1-r}$$

$$= (n+1)n(n-1)\ldots(n+1+1-r)$$

$$= P(n+1, r).$$
25) \[ |S| = n \]

Count \[ \# \left\{ (A, B) \mid A \subseteq B \subseteq S \right\} \].

**Proof:**

If \( A \subseteq B \subseteq S \),

then \( S \) is partitioned into 3 disjoint subsets:

\[ A \cup B \cup (S - B). \]

(Since \( A \cup (B - A) = B \) and \( B \cup (S - B) = S \), so \( A \cup (B - A) \cup (S - B) = S \) and \( A \cap (B - A) = \emptyset \), \( A \cap (S - B) = \emptyset \), and \( B \cap (S - B) = \emptyset \).)

So choosing subsets \( A \subseteq B \subseteq S \) is the same as choosing which of \( A, B - A, \) and \( S - B \) to place each \( x \in S \) into.

Result: \[ 3^n \].
27) \( n, r \in \mathbb{Z} \), \( 1 \leq r \leq n \).

Show

\[
\binom{n-r}{n-r} = \binom{n+2}{r+1} - 2 \binom{n+1}{r+1} + \binom{n}{r+1}
\]

Combination proof outline

LHS: \( \# \{(r-1) \text{ subsets of size-} n \text{ set}\} \)

RHS: Let \( S \) be a set of size \( n \),

Let \( x, y \in S \), so that \( |S \cup \{x, y\}| = n + 2 \).

So a size \( r-1 \) subset of \( S \) is the same as a size \( r+1 \) subset of \( S \cup \{x, y\} \) that contains \( x \) or \( y \).

There are \( \binom{n-r}{r+1} \) (subsets of \( S \cup \{x, y\} \) that contain \( x \) or \( y \).

Total:

\[
\begin{align*}
&= \binom{n+2}{r+1} - 2 \binom{n+1}{r+1} + \binom{n}{r+1}.
\end{align*}
\]
Use induction to show
\[ \sum_{j=2}^{n} C(j,2) = C(n+1,3) \] (*)

Proof: Let \( P(n) \) be (*)

Base case: \( P(2) \) says
\[ \sum_{j=2}^{2} C(j,2) = C(2,2) = 1 = C(2+1,3). \] \( \checkmark \)

Goal: Assume \( P(n) \) for fixed \( n \geq 2 \) and show \( P(n+1) \) which is
\[ \sum_{j=2}^{n+1} C(j,2) = C(n+2,3). \]

Inductive step: Assume \( P(n) \) for fixed \( n \geq 2 \).

Then
\[ \sum_{j=2}^{n+1} C(j,2) = \left( \sum_{j=2}^{n} C(j,2) \right) + C(n+1,2) \]
\[ = C(n+1,3) + C(n+1,2) \]
\[ = C(n+2,3) \] by Pascal's identity.

Conclusion: Since \( P(2) \) holds and \( P(n) \) implies \( P(n+1) \) for \( n \geq 2 \), we have \( P(k) \) holds for \( k \geq 2 \) (\( k \in \mathbb{Z}_{\geq 2} \)).
29) Show
\[ \sum_{k=0}^{n} 3^k \binom{n}{k} = 4^n \quad \text{for } n \in \mathbb{Z}. \]

Note: For \( n < 0 \), this does not hold!
Prove for \( n > 0 \).

Proof: Binomial Thm says
\[ (x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}. \]
Plug in \( x = 3 \), \( y = 1 \) to get
\[ 4^n = (3+1)^n = \sum_{k=0}^{n} \binom{n}{k} 3^k 1^{n-k} = \sum_{k=0}^{n} \binom{n}{k} 3^k. \]

(For a combinatorial pf, count # ways to answer a multiple choice exam w/ 3 choices per question.
LHS: let \( k = \# \) left blank
RHS: just count.)
33. If exactly 2 01's, that means (maybe some 1's)
    (some 0's)
    (some 1's)
    (some 0's)
    (some 1's).

Cases:

starts w/ 1
(i.e. 01-01-01-01-01)

\[
\frac{1111010001}{1101}
\]

\[
\frac{1110110001}{2^{nd}01}
\]

n stars
4 bars
at least one star per region:
\[
\binom{n-5+4}{4}
\]

(f or example,
***|*|***|**
means
\[
\frac{1111010001}{1101}
\]

starts w/ 0
(i.e. 00...01-10-01-01...

\[
\frac{000...01}{1101}
\]

\[
\frac{000...01}{2^{nd}01}
\]

n stars
3 bars
at least one star per region:
\[
\binom{n-4+3}{3}
\]

(f or example,
**|****|*|***
means
\[
\frac{000...01}{1101}
\]

Total:
\[
\binom{n-1}{4} + \binom{n-1}{3} = \binom{n}{4}
\]

Ah nah!

Alternatively:
Start w/ 1. Of the n bits, pick 4 places to switch bit type (ok to switch @ beginning, but not @ end; don't switch more than 1 once per bit).

\[
\binom{n}{4}
\]

ex: \[---\uparrow\uparrow\uparrow\uparrow\] means 0000100011
35) \(\left(\frac{20}{8}\right)^3 \left(\frac{20-3}{4}\right)^4 \left(\frac{20-8-3-4}{5}\right)\)
\[= \frac{20!}{8!3!4!5!}\]

36) \(\frac{6!}{6!} = 5!\)

37) \(5^{24}\)

38) \(\binom{(12-3-3)+(3-1)}{3-1}\)

39) (a) \(\binom{(17-(1+2+3))+(3-1)}{(3-1)}\)

(b) \[\sum_{x_1=0}^{5} \binom{(17-x_1)-5}{2-1} = \sum_{x_1=0}^{5} (13-x_1)\]
\[= 6 \cdot 13 - \sum_{x_1=1}^{5} x_1.\]

(c) Count total \(x_3 > 5\) - \# \{ \(x_1 > 4, x_2 > 3\) \}
\[\binom{17-5+(3-1)}{3-1} - \binom{17-(5+3+2)+(3-1)}{3-1}\]

\(\)
40) (a) \[ \#D = 3 \quad \#E = 2 \quad \#R = 2 \quad \#C = 1 \quad \#O = 1 \quad \#N = 1 \]
\[ \frac{(3 + 2 + 2 + 1 + 1 + 1)!}{3! \cdot 2! \cdot 2! \cdot 1! \cdot 1! \cdot 1!} \]

(b) \[ \frac{(1 + 2 + 2 + 1 + 1 + 1)!}{1! \cdot 2! \cdot 2! \cdot 1! \cdot 1! \cdot 1!} \]

(c) \[ \frac{(1 + 2 + 2 + 1 + 1 + 1)!}{1! \cdot 2! \cdot 2! \cdot 1! \cdot 1! \cdot 1!} \quad (\text{think of 'PPP' as one letter}) \]

41) (a) \[ \sum_{k=0}^{4} \binom{10}{k} \]

(b) \[ \sum_{k=0}^{10} \binom{10}{k} \]

(c) \[ \sum_{k=0}^{10} \binom{10}{k} \]

42) \[ 26 \cdot [3 + 3 + 3! \cdot 8] \]
\[ \frac{3! \times 3 \times 3}{\text{permuted}} \quad \frac{3! \times 3 \times 3}{\text{permuted}} \quad \frac{3! \times 3 \times 3}{\text{permuted}} \]

43) \[ \binom{n-m + m-1}{m-1} = \binom{n-1}{m-1} \]

44) Think: 8 girls = stars 6 boys = bars \[ \text{we want at least one star between each pair of bars} \]
\[ \binom{8-6+1}{6} = \binom{9}{6} \]
then pick labels of 1's \( \text{w/ n boys} \) \( \text{it's w/ specific girls} \)

\[ \text{total: } \binom{9}{6} \cdot 8! \cdot 6! \]