Math 365 – WORKSHEET – Wednesday 3/27/19
See last few pages for some review!

1. Write the generating functions for the following sequences, in both their series form and closed form (the simplified form). Assume in each case that the sequence starts at a_0. For example, the sequence $1, 2, 3, 4, 4, 4, 4, \ldots$ has the generating function

$$\underbrace{1 + 2x + 3x^2 + \sum_{n=3}^{\infty} 4x^n}_{\text{series form}} = 1 + 2x + 3x^2 + 4x^3 \sum_{n=0}^{\infty} x^n = \underbrace{1 + 2x + 3x^2 + \frac{4x^3}{1-x}}_{\text{closed form}}.$$

Start by writing the *sequence itself* in a closed form. For example, the above sequence is $a_0 = 1, a_1 = 2, a_2 = 3,$ and $a_n = 4$ for $n \geq 3$.

(a) $5, 5, 5, 5, 5, \ldots$

$$a_n = 5 \text{ for } n \geq 0$$

$$G(x) = a_0 + a_1 x + a_2 x^2 + \cdots = 5 + 5x + 5x^2 + \cdots$$
$$= \sum_{n=0}^{\infty} 5x^n = 5 \sum_{n=0}^{\infty} x^n = \frac{5}{1-x}$$

(b) $1, 3, 9, 27, \ldots$

$$a_n = 3^n \text{ for } n \geq 0$$

$$G(x) = a_0 + a_1 x + a_2 x^2 + \cdots = 1 + 3x + 3^2 x^2 + \cdots$$
$$= \sum_{n=0}^{\infty} 3^n x^n = \sum_{n=0}^{\infty} (3x)^n = \frac{1}{1-3x}$$

(c) $1, -1, 1, -1, 1, -1, \ldots$

$$a_n = (-1)^n \text{ for } n \geq 0$$

$$G(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = 1 + (-1)x + x^2 + (-1)x^3 + \cdots$$
$$= \sum_{n=0}^{\infty} (-1)^n x^n = \sum_{n=0}^{\infty} (-x)^n = \frac{1}{1+x}$$

(d) $1, -2, 4, -8, 16, \ldots$

$$a_n = (-2)^n \text{ for } n \geq 0$$

$$G(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = 1 + (-2)x + (-2)^2 x^2 + (-2)^3 x^3 + \cdots$$
$$= \sum_{n=0}^{\infty} (-2)^n x^n = \sum_{n=0}^{\infty} (-2x)^n = \frac{1}{1+2x}$$

(e) $1, 0, 0, 1, 1, 1, 1, 1, \ldots$

$$a_0 = 1, a_1 = a_2 = 0, a_n = 1 \text{ for } n \geq 1$$

$$G(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots = 1 + 0x + 0x^2 + x^3 + x^4 + \cdots$$
$$= 1 + \sum_{n=3}^{\infty} x^n = 1 + x^3 \sum_{n=0}^{\infty} x^n = 1 + \frac{x^3}{1-x}$$

(f) $0, 0, 0, 2, 2, 2, 2, 2, \ldots$

$$a_0 = a_1 = a_2 = 0, a_n = 2 \text{ for } n \geq 3$$

$$G(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots = 0 + 0x + 0x^2 + 2x^3 + 2x^4 + \cdots$$
$$= \sum_{n=3}^{\infty} 2x^n = 2x^3 \sum_{n=0}^{\infty} x^n = \frac{2x^3}{1-x}$$

(g) $1, 3, -2, 5, 10, 20, 40, 80, \ldots$

$$a_0 = 1, a_1 = 3, a_2 = -2, a_3 = 5, a_n = 10 * (2)^{n-4} \text{ for } n \geq 4,$$

$$G(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots = 1 + 3x + (-2)x^2 + 5x^3 + 10x^4 + 20x^5 + \cdots$$
$$= 1 + 3x - 2x^2 + 5x^3 \sum_{n=0}^{\infty} 2^n x^n = 1 + 3x - 2x^2 + \frac{5x^3}{(1-2x)}$$

(h) $1, 2, 3, 4, 5, \ldots$

$$a_n = n + 1 \text{ for } n \geq 0, \quad \textstyle\sum_{n=0}^{\infty} (n+1)x^n = \frac{1}{(1-x)^2}$$

$$G(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots = 1 + 2x + 3x^2 + 4x^3 + \cdots$$
$$= \sum_{n=0}^{\infty} (n+1)x^n = \frac{1}{(1-x)^2}$$

(i) $3, 6, 9, 12, 15, \ldots$

$$a_n = 3(n+1) \text{ for } n \geq 0$$

$$G(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots = 3 + 6x + 9x^2 + 12x^3 + \cdots$$
$$= \sum_{n=0}^{\infty} 3(n+1)x^n = \frac{3}{(1-x)^2}$$

(j) $2, 6, 12, 20, 30, 42, \ldots$ (hint: $6 = 3 * 2, 12 = 4 * 3, 20 = 5 * 4, \ldots$)

$$a_n = (n+2)(n+1) \text{ for } n \geq 0$$

$$G(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = 2 + 6x + 12x^2 + 20x^3 + \cdots$$
$$= \sum_{n=0}^{\infty} (n+2)(n+1)x^n = 2\sum_{n=0}^{\infty} \frac{(n+2)(n+1)}{2} x^n = \frac{2}{(1-x)^3}$$

(k) $1, 0, -2, 2, 6, 12, 20, 30, 42, \ldots$

$$a_0 = 1, a_1 = 0, a_2 = -2, a_n = (n-1)(n-2) \text{ for } n \geq 3,$$

$$G(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots = 1 + 0x + (-2)x^2 + 2x^3 + 6x^4 + \cdots$$
$$= 1 - 2x^2 + \sum_{n=3}^{\infty} (n-1)(n-2)x^n = 1 - 2x^2 + x^3 \sum_{n=0}^{\infty} (n+2)(n+1)x^n = 1 - 2x^2 + \frac{2x^3}{(1-x)^3}$$

2. For each of the following, write the generating function for which the coefficient of x^r is the answer.
   (a) How many ways can postage of r cents can be pasted on an envelope using 3-cent, 4-cent, and 20-cent stamps, if the arrangement of the postage doesn't matter?

$$\begin{aligned}(1+x^3+x^6+\cdots)&(1+x^4+x^8+\cdots)(1+x^{20}+x^{40}+\cdots)\\ &=\left(\frac{1}{1-x^3}\right)\left(\frac{1}{1-x^4}\right)\left(\frac{1}{1-x^{20}}\right)\end{aligned}$$

   (b) How many ways can you make change for \$100 using \$5, \$10, \$20, and \$50 bills?

$$(1+x^5+x^{10}+\cdots+x^{100})(1+x^{10}+x^{20}+\cdots+x^{100})(1+x^{20}+x^{40}+\cdots+x^{100})(1+x^{50}+x^{100})$$

   (c) How many ways can you make change for \$100 using \$5, \$10, \$20, and \$50 bills, if you only have 4 of each kind of bill?

$$(1+x^5+x^{10}+x^{15}+x^{20})(1+x^{10}+x^{20}+x^{30}+x^{40})(1+x^{20}+x^{40}+\cdots+x^{80})(1+x^{50}+x^{100})$$

   (d) How many ways can you make change for \$100 using \$5, \$10, \$20, and \$50 bills, if you have to use at least one of each kind of bill, but you only have one \$50?

$$(x^5+x^{10}+\cdots+x^{100})(x^{10}+x^{20}+\cdots+x^{100})(x^{20}+x^{40}+\cdots+x^{100})(x^{50})$$

3. Suppose you have 5 pennies, 3 nickels, and a dime in your pocket. When reaching in to your pocket to pull out change, you do so totally randomly (any subset of coins is equally likely).
   (a) What is the generating function for the total value of any set of change you might pull out?

$$(1+x+x^2+x^3+x^4+x^5)(1+x^5+x^{10}+x^{15})(1+x^{10})$$

   (b) What value(s) of change are you most likely to pull out?
   [Hint: use something like wolframalpha.com to expand your polynomial]

$$\begin{aligned}(1+x+x^2+&x^3+x^4+x^5)(1+x^5+x^{10}+x^{15})(1+x^{10})\\ &=x^{30}+x^{29}+x^{28}+x^{27}+x^{26}+2x^{25}+x^{24}+x^{23}+x^{22}+x^{21}+3x^{20}\\ &\quad+2x^{19}+2x^{18}+2x^{17}+2x^{16}+\boxed{4x^{15}}+2x^{14}+2x^{13}+2x^{12}+2x^{11}\\ &\quad+3x^{10}+x^9+x^8+x^7+x^6+2x^5+x^4+x^3+x^2+x+1\end{aligned}$$

Since x^{15} has the largest coefficient, 15 cents is the most likely denomination.

4. For each of the following recursion relations, suppose that $G(x)$ is the generating function for the corresponding solution. Write an expression in terms of $G(x)$ and other familiar generating functions that will allow you to solve for $G(x)$. For example, if

$$a_n = 8a_{n-1} + 10^{n-1}, \quad \text{then} \quad \boxed{G(x) = a_0 + 8xG(x) + \frac{x}{1-10x}.}$$

The relevant calculation here is

$$\begin{aligned} G(x) &= \sum_{n=0}^{\infty} a_n x^n \qquad = a_0 + \sum_{n=1}^{\infty} a_n x^n \quad = a_0 + \sum_{n=1}^{\infty} (8a_{n-1} + 10^{n-1}) x^n \\ &= a_0 + 8\sum_{n=1}^{\infty} a_{n-1} x^n + \sum_{n=1}^{\infty} 10^{n-1} x^n \quad = a_0 + 8\sum_{n=0}^{\infty} a_n x^{n+1} + \sum_{n=0}^{\infty} 10^n x^{n+1} \\ &= a_0 + 8xG(x) + x\sum_{n=0}^{\infty} 10^n x^n \quad = a_0 + 8xG(x) + x\frac{1}{1-10x}. \end{aligned}$$

(See p. 7 for another example)

(a) $a_n = 7a_{n-1}$

$$\begin{aligned} G(x) &= \sum_{n=0}^{\infty} a_n x^n \qquad = a_0 + \sum_{n=1}^{\infty} a_n x^n \quad = a_0 + \sum_{n=1}^{\infty} 7a_{n-1} x^n \\ &= a_0 + 7x\sum_{\ell=0}^{\infty} a_\ell x^\ell \quad = a_0 + 7xG(x). \end{aligned}$$

So

$$\boxed{G(x) = a_0 + 7xG(x).}$$

(b) $a_n = a_{n-1} + a_{n-2}$

$$\begin{aligned} G(x) &= \sum_{n=0}^{\infty} a_n x^n \qquad = a_0 + a_1 x + \sum_{n=2}^{\infty} a_n x^n \quad = a_0 + a_1 x + \sum_{n=2}^{\infty} (a_{n-1} + a_{n-2}) x^n \\ &= a_0 + a_1 x + \sum_{n=2}^{\infty} a_{n-1} x^n + \sum_{n=2}^{\infty} a_{n-2} x^n \\ &= a_0 + a_1 x + x\sum_{n=1}^{\infty} a_n x^n + x^2 \sum_{n=0}^{\infty} a_n x^n \\ &= a_0 + a_1 x + x(G(x) - a_0) + x^2 G(x). \end{aligned}$$

So

$$\boxed{G(x) = a_0 + a_1 x + x(G(x) - a_0) + x^2 G(x).}$$

(c) $a_n = a_{n-1} + 2a_{n-2}$

$$G(x) = \sum_{n=0}^{\infty} a_n x^n \qquad = a_0 + a_1 x + \sum_{n=2}^{\infty} a_n x^n \quad = a_0 + a_1 x + \sum_{n=2}^{\infty} (a_{n-1} + 2a_{n-2}) x^n$$
$$= a_0 + a_1 x + \sum_{n=2}^{\infty} a_{n-1} x^n + 2 \sum_{n=2}^{\infty} a_{n-2} x^n$$
$$= a_0 + a_1 x + x \sum_{n=1}^{\infty} a_n x^n + 2x^2 \sum_{n=0}^{\infty} a_n x^n$$
$$= a_0 + a_1 x + x(G(x) - a_0) + 2x^2 G(x).$$

So

$$\boxed{G(x) = a_0 + a_1 x + x(G(x) - a_0) + 2x^2 G(x).}$$

(d) $a_n = a_{n-1} + 2a_{n-2} + 2^n$

$$G(x) = \sum_{n=0}^{\infty} a_n x^n \qquad = a_0 + a_1 x + \sum_{n=2}^{\infty} a_n x^n \quad = a_0 + a_1 x + \sum_{n=2}^{\infty} (a_{n-1} + 2a_{n-2} + 2^n) x^n$$
$$= a_0 + a_1 x + \sum_{n=2}^{\infty} a_{n-1} x^n + 2 \sum_{n=2}^{\infty} a_{n-2} x^n + \sum_{n=2}^{\infty} 2^n x^n$$
$$= a_0 + a_1 x + x \sum_{n=1}^{\infty} a_n x^n + 2x^2 \sum_{n=0}^{\infty} a_n x^n + 4x^2 \sum_{n=0}^{\infty} 2^n x^n$$
$$= a_0 + a_1 x + x(G(x) - a_0) + 2x^2 G(x) + \frac{4x^2}{1 - 2x}.$$

So

$$\boxed{G(x) = a_0 + a_1 x + x(G(x) - a_0) + 2x^2 G(x) + \tfrac{4x^2}{1-2x}.}$$

(e) $a_n = a_{n-1} + 2a_{n-2} + n$

$$\begin{aligned} G(x) = \sum_{n=0}^{\infty} a_n x^n \qquad &= a_0 + a_1 x + \sum_{n=2}^{\infty} a_n x^n \quad = a_0 + a_1 x + \sum_{n=2}^{\infty} (a_{n-1} + 2a_{n-2} + n) x^n \\ &= a_0 + a_1 x + \sum_{n=2}^{\infty} a_{n-1} x^n + 2 \sum_{n=2}^{\infty} a_{n-2} x^n + \sum_{n=2}^{\infty} n x^n \\ &= a_0 + a_1 x + x \sum_{n=1}^{\infty} a_n x^n + 2x^2 \sum_{n=0}^{\infty} a_n x^n + 2x^2 \sum_{n=0}^{\infty} (n+1) x^n \\ &= a_0 + a_1 x + x(G(x) - a_0) + 2x^2 G(x) + \frac{2x^2}{(1-x)^2}. \end{aligned}$$

So

$$\boxed{G(x) = a_0 + a_1 x + x(G(x) - a_0) + 2x^2 G(x) + \tfrac{2x^2}{(1-x)^2}.}$$

(f) $a_n = a_{n-1} + 2a_{n-2} + 2^n + n + 7.$

$$\begin{aligned} G(x) = \sum_{n=0}^{\infty} a_n x^n \qquad &= a_0 + a_1 x + \sum_{n=2}^{\infty} a_n x^n \quad = a_0 + a_1 x + \sum_{n=2}^{\infty} (a_{n-1} + 2a_{n-2} + 2^n + n + 7) x^n \\ &= a_0 + a_1 x + \sum_{n=2}^{\infty} a_{n-1} x^n + 2 \sum_{n=2}^{\infty} a_{n-2} x^n + \sum_{n=2}^{\infty} 2^n x^n + \sum_{n=2}^{\infty} n x^n + \sum_{n=2}^{\infty} 7 x^n \\ &= a_0 + a_1 x + x \sum_{n=1}^{\infty} a_n x^n + 2x^2 \sum_{n=0}^{\infty} a_n x^n + 4x^2 \sum_{n=0}^{\infty} 2^n x^n + 2x^2 \sum_{n=0}^{\infty} (n+1) x^n + 7x^2 \sum_{n=0}^{\infty} x^n \\ &= a_0 + a_1 x + x(G(x) - a_0) + 2x^2 G(x) + \frac{4x^2}{1-2x} + \frac{2x^2}{(1-x)^2} + \frac{7x^2}{1-x}. \end{aligned}$$

So

$$\boxed{G(x) = a_0 + a_1 x + x(G(x) - a_0) + 2x^2 G(x) + \tfrac{4x^2}{1-2x} + \tfrac{2x^2}{(1-x)^2} + \tfrac{7x^2}{1-x}.}$$

5. Use your answer to part (b) of the previous problem to solve for a_n when $a_n = a_{n-1} + a_{n-2}$, $a_0 = 0$ and $a_1 = 1$. Compare this the solution we found in Section 8.2 (example 4 in the book).

From part (b), $G(x) = a_0 + a_1 x + x(G(x) - a_0) + x^2 G(x)$. So

$$G(x)\left(1 - x - x^2\right) = a_0 + a_1 x - a_0 x = 0 + 1 * x - 0 * x = x,$$

and so $$G(x) = -\frac{x}{x^2 + x - 1} = x\left(\frac{1}{((-1+\sqrt{5})/2 - x)((-1-\sqrt{5})/2 - x)}\right).$$

Notice that

$$\frac{1}{a - x} = \frac{1}{a(1 - \frac{1}{a}x)} = \frac{1}{a}\frac{1}{(1 - \frac{1}{a}x)}.$$

Also

$$\frac{1}{(-1-\sqrt{5})/2} = -\frac{2}{1+\sqrt{5}} = -\frac{2(1-\sqrt{5})}{(1+\sqrt{5})(1-\sqrt{5})} = -\frac{2(1-\sqrt{5})}{1-5} = \frac{1+\sqrt{5}}{2}$$

and

$$\frac{1}{(-1+\sqrt{5})/2} = -\frac{2}{1-\sqrt{5}} = -\frac{2(1+\sqrt{5})}{(1-\sqrt{5})(1+\sqrt{5})} = -\frac{2(1+\sqrt{5})}{1-5} = \frac{1-\sqrt{5}}{2}.$$

So

$$\begin{aligned}
G(x) &= x\left(\frac{1}{((-1+\sqrt{5})/2 - x)((-1-\sqrt{5})/2 - x)}\right) \\
&= x\left(\frac{(2/(-1+\sqrt{5}))(2/(-1-\sqrt{5}))}{(1 - 2/(-1+\sqrt{5})x)(1 - 2/(-1-\sqrt{5})x)}\right) \\
&= x\left(\frac{(-(1-\sqrt{5})/2)(-(1+\sqrt{5})/2)}{(1 - ((1-\sqrt{5})/2)x)(1 - ((1-\sqrt{5})/2)x)}\right) \\
&= -x\left(\frac{1}{(1 - ((1-\sqrt{5})/2)x)(1 - ((1-\sqrt{5})/2)x)}\right) \\
&= -\left(\frac{x}{(1-\varphi x)(1-\bar{\varphi}x)}\right)
\end{aligned}$$

since $((1-\sqrt{5})/2)((1+\sqrt{5})/2) = (1-5)/4 = -1$, where

$$\varphi = \frac{1-\sqrt{5}}{2} \qquad \text{and} \qquad \bar{\varphi} = \frac{1+\sqrt{5}}{2}.$$

Using the method of partial fractions, solve

$$\frac{-x}{(1-\varphi x)(1-\bar{\varphi}x)} = \frac{A}{(1-\varphi x)} + \frac{B}{(1-\bar{\varphi}x)},$$

i.e.

$$\begin{aligned}
-x &= A(1-\bar{\varphi}x) + B(1-\varphi x)) \\
&= -(A\bar{\varphi} + B\varphi)x + (A + B)
\end{aligned}$$

So

$$\text{coef of 1: } 0 = A + B, \qquad \text{so } B = -A; \text{ and}$$

$$\text{coef of } x\text{: } -1 = -(A\bar{\varphi} + B\varphi) = -A\frac{1-\sqrt{5}}{2} + A\frac{1+\sqrt{5}}{2} = A\sqrt{5}$$
$$\text{so } A = -1/\sqrt{5} \text{ and } B = 1/\sqrt{5}.$$

Therefore,

$$\begin{aligned} G(x) = \frac{x}{1-x-x^2} &= -\left(\frac{x}{(1-\varphi x)(1-\bar{\varphi}x)}\right) \\ &= \frac{1}{\sqrt{5}}\left(\frac{1}{(1-\bar{\varphi}x)} - \frac{1}{(1-\varphi x)}\right) \\ &= \frac{1}{\sqrt{5}}\left(\sum_{n=0}^{\infty}\bar{\varphi}^n x^n - \sum_{n=0}^{\infty}\varphi^n x^n\right) \\ &= \frac{1}{\sqrt{5}}\sum_{n=0}^{\infty}(\bar{\varphi}^n - \varphi^n)x^n \end{aligned}$$

So

$$a_n = \frac{\bar{\varphi}^n - \varphi^n}{\sqrt{5}} = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right).$$

This is exactly what we saw in Example 4 of Section 8.2.