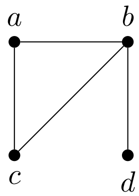
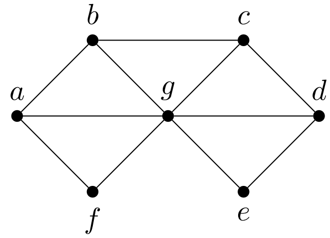
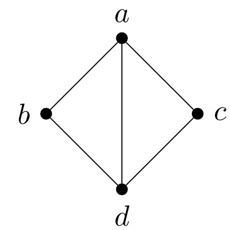
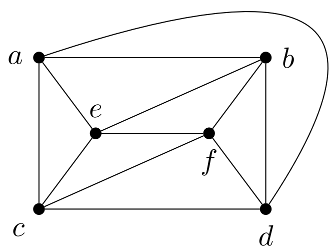
Math 365 – Monday 4/29/19

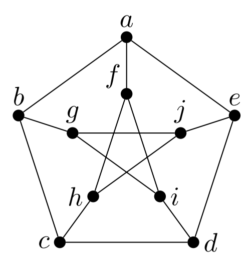
Exercise 55. Let A, B, C, D, G, and H be the graphs

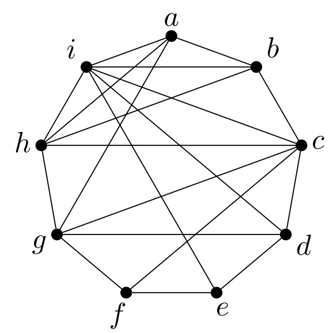
$A =$ 

$B =$ 

$C =$ 

$D =$ 

$G =$ 

$H =$ 

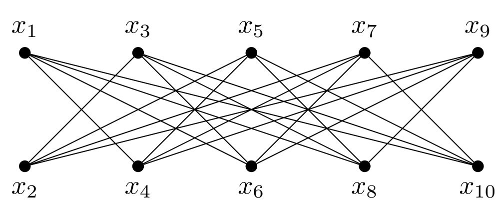
(a) Calculate the chromatic numbers for A, B, C, D, G, and H. For each, give an example of a vertex coloring of the corresponding graph using exactly χ colors.
(b) Which of A, B, C, D, G, and H have the property that removing a single vertex will reduce the chromatic number?
(c) Classify all graphs with chromatic number (i) 1, and (ii) 2.
(d) What are the chromatic numbers of

(i) K_n, (ii) $K_{m,n}$, (iii) C_n, (iv) W_n, (v) Q_n?

Exercise 56. (a) What are the clique and independence numbers of A, B, C, D, G, and H from the previous problem? How do ω and $|V|/\alpha$ compare to χ for each graph?
(b) What are the clique and independence numbers of

(i) K_n, (ii) $K_{m,n}$, (iii) C_n, (iv) W_n, (v) Q_n?

How do ω and $|V|/\alpha$ compare to χ for each graph? (You may need to break into cases.)
(c) Explain why the clique number of the complement of a bipartite is no smaller than the number of vertices in each part. (Recall that the *parts* of a bipartite graph are the two collections of pairwise non-adjacent vertices.)
(d) Notice that the graph

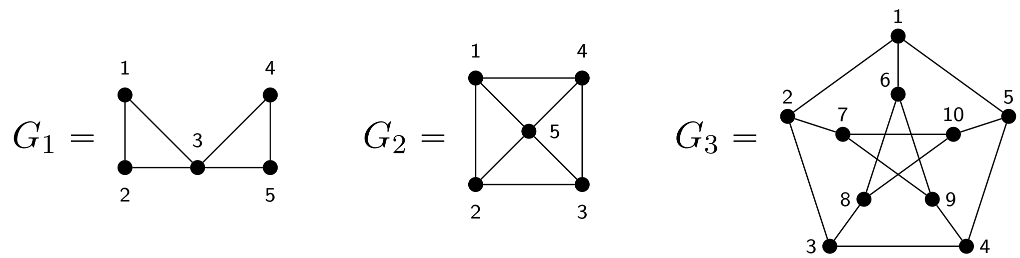
$G =$ 

is bipartite, so should have chromatic number 2. Now color this graph using the so-called "greedy algorithm": name your colors "color 1, color 2, …". First color x_1 with color 1; then color x_2 with the lowest color possible (i.e. color 1 if you can, but color 2 if you can't); then color x_3 with the lowest color possible; and so on. How many colors did you need? What is a better way to color G?

Welcome back warmup

Recall, a graph invariant is a statistic about a graph that is preserved under isomorphisms (relabeling of the vertices).

Before looking through the notes, list as many "graph invariants" as you can from memory. Then, for each, compute that invariant on the following graphs.



Once you run out, there is a list of graph invariants on the next page–fill out your list and continue computing graph invariants.

Graph invariants

Recall, a graph invariant is a statistic about a graph that is preserved under isomorphisms (relabeling of the vertices).

1. $|V|$, $|E|$
2. Degree sequence

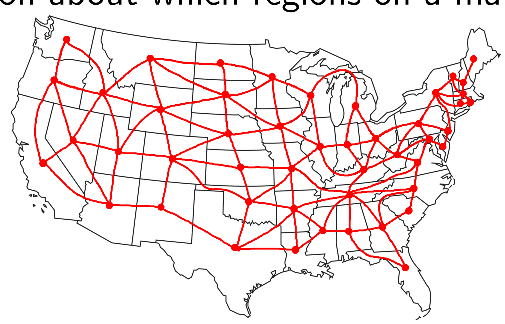
   Also: Minimum degree, maximum degree, vertex of degree d_1 adjacent to vertex of degree d_2, ...
3. Bipartite or not

   If any subgraph is not bipartite, then G is not bipartite. A graph is bipartite if and only if it has no odd cycles as subgraphs.
4. Paths or cycles of particular lengths

   Also: longest path or cycle length, maximal paths of certain lengths, ...
5. Edge connectivity $\lambda(G)$ and vertex connectivity $\kappa(G)$.
6. Does it have an Euler trail/circuit?
7. Does it have a Hamilton path/cycle?

Maps coloring

Encode information about which regions on a map share a border.

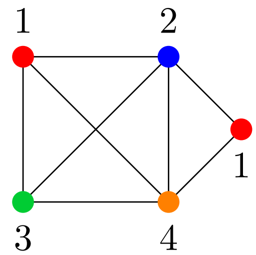


Mark each region with a vertex. Draw an edge between to vertices if the corresponding regions share a boarder.

Map coloring problems: color a map so that no two adjacent regions get the same color.

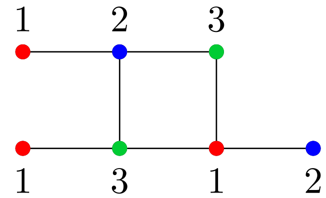
Four color problem: In order to color the vertices of a plane map so that no two adjacent vertices get the same color, you will need no more than four colors. (No elementary proof!)

A coloring of a simple graph is the assignment of a color to each vertex of the graph so that no two adjacent vertices are assigned the same color.

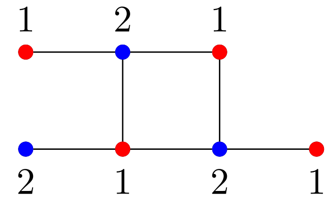


using 4 colors

$\chi = 4$



using 3 colors



using 2 colors

$\chi = 2$

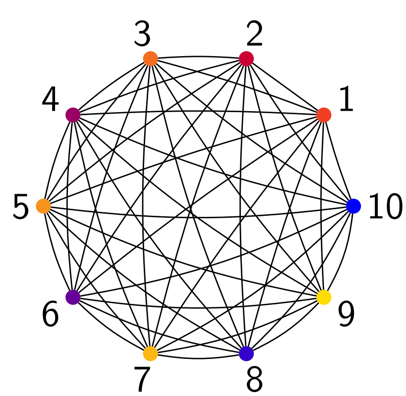
The chromatic number of a graph G, denoted $\chi(G)$, is the least number of colors needed for a coloring of this graph.

To calculate, argue that the graph can't be colored in $\chi - 1$ colors, and then give a coloring with exactly χ colors.

A coloring of a simple graph is the assignment of a color to each vertex of the graph so that no two adjacent vertices are assigned the same color. The chromatic number of a graph G, denoted $\chi(G)$, is the least number of colors needed for a coloring of this graph.

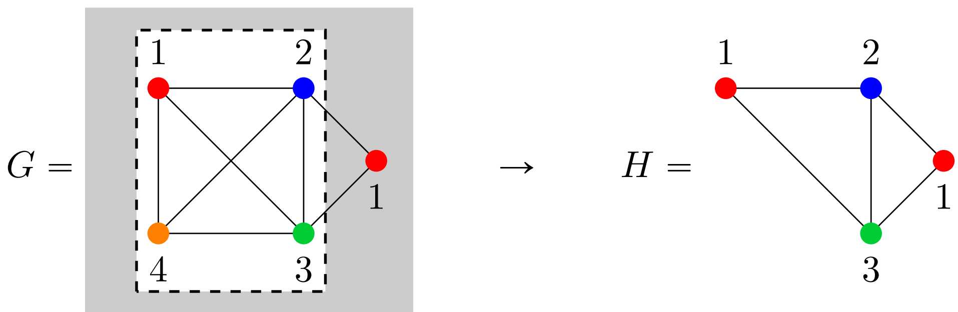
Lemma

The complete graph on n vertices can only be colored in exactly n colors. Namely $\chi(K_n) = n$.



Cliques

Notice: If $H \subseteq G$ are graphs, a good coloring of G restricts to a good coloring of H:



So $\boxed{\chi(G) \geqslant \chi(H)}$.

Strategy: Look for complete graphs as subgraphs, since they have high chromatic numbers!

Ex: the G has K_4 as a subgraph, so $\chi(G) \geqslant \chi(K_4) = 4$. And since we gave a 4-coloring, we know $\chi(G) \leqslant 4$. So $\boxed{\chi(G) = 4}$ exactly.

Cliques

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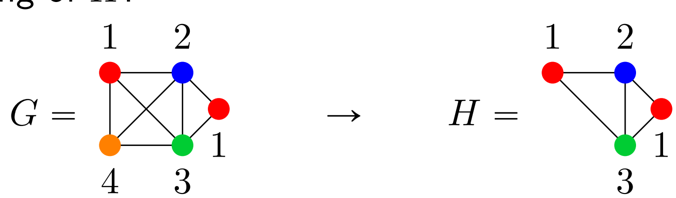


So $\boxed{\chi(H) \leqslant \chi(G)}$.

Strategy: Look for complete graphs as subgraphs, since they have high chromatic numbers!

We call a complete subgraph of a graph G (a set of vertices that are all adjacent to each other) a clique. The clique number $\omega(G)$ of G is the size of a biggest clique.

Ex: $\omega(G) = 4$, $\omega(H) = 3$.

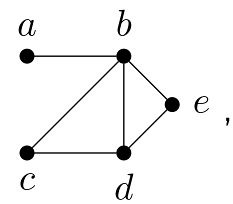
Then

So $\boxed{\omega(G) \leqslant \chi(G)}$.

(Both chromatic and clique numbers are graph invariants.)

We call a complete subgraph of a graph G (a set of vertices that are all adjacent to each other) a clique. The clique number $\omega(G)$ of G is the size of a biggest clique. So $\boxed{\omega(G) \leqslant \chi(G)}$.

Similarly, define and independent set of G as set of vertices that have no edges between them.
Ex: In

$G =$ 

, $\boxed{\alpha(H) = 3}$

$\{a, d\}$ is an independent set; so is $\{c, e\}$; so is $\{a, c, e\}$. But $\{a, d, e\}$ is not, since d and e share an edge.

The independence number $\alpha(G)$ is the size of a largest independent set. (Yet another graph invariant!)

Note: $\alpha(G) = \omega(\bar{G})$
(Recall $\bar{G}$ is the *complement* of G — toggle all of the edges.)

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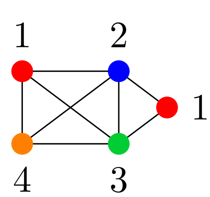
Similarly, define and independent set of G as set of vertices that have no edges between them. The independence number $\alpha(G)$ is the size of a largest independent set. (Yet another graph invariant!)

Note: $\alpha(G) = \omega(\bar{G})$
(Recall $\bar{G}$ is the *complement* of G — toggle all of the edges.)

Also, in a coloring of a graph, the vertices of any given color are all independent.



So $\boxed{\chi(G) \geqslant |V|/\alpha(G) \text{ and } \chi(G) \geqslant \omega(G)}$

(Dividing $|V|$ vertices evenly into sets of size $\alpha(G)$ gets you $|V|/\alpha(G)$ sets.)