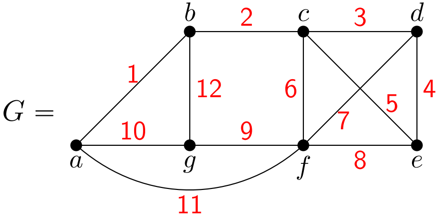
Warmup:

Let



(i) Verify that G is connected by giving an example of a walk from vertex a to each of the vertices b–g.

(ii) What is the shortest path from a to c? to e?

(iii) What is the longest path from a to c? to e?

(iv) What is a longest path in G?

(v) Does G have any maximal paths that are shorter than the path in part (iv)?

(Recall a path is a walk with no repeated vertices or edges, and a maximal path is one that can't be extended in either direction.)

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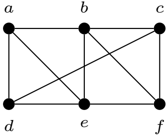
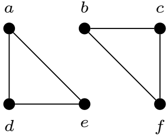
$G =$ 

$H =$ 

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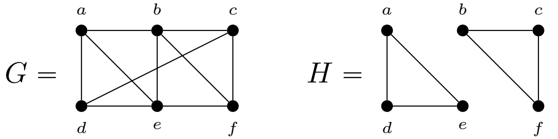

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"Connected" is an equivalence relation on vertices: we say $u \sim v$ if there is a walk from u to v. A connected component of a graph is a maximally connected subgraph of G (H above has two connected components), i.e. the equivalence classes under the connectedness relation.

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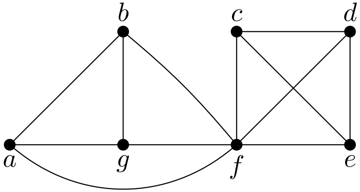
$G =$ 

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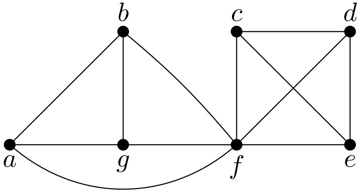
$G =$ 

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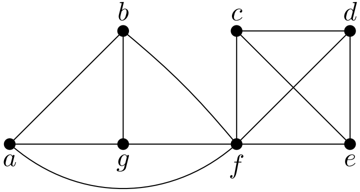

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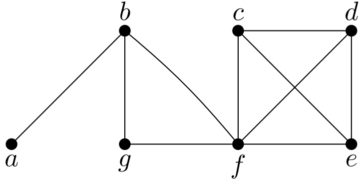

the cut vertices are b and f, and edge $a{-}b$ is the only cut edge.

You try:

Identify the cut edges and vertices (if any) of the following graphs:

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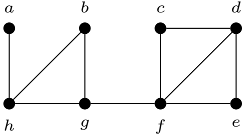

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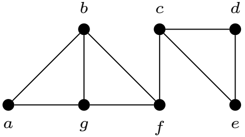

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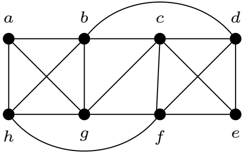

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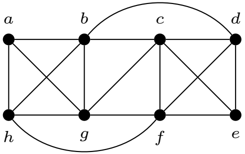
$K =$ 

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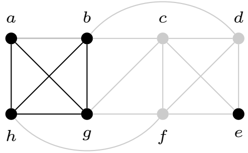
$K =$ 

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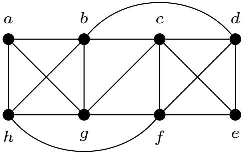

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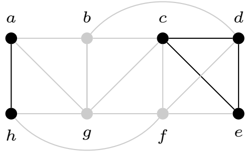
$K =$ 

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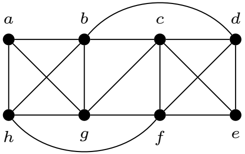

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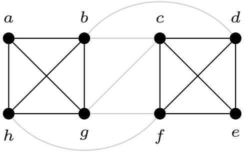
$K =$ 

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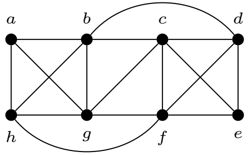
$K =$ 

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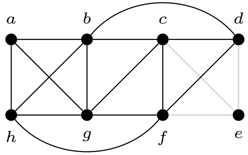

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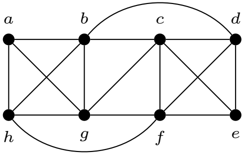
$K =$ 

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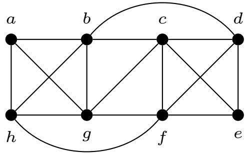
$K =$ 

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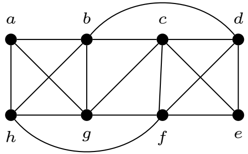
$G =$ 

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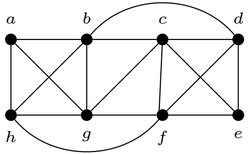

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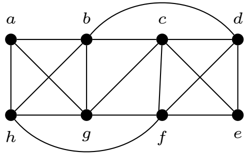
$G =$ 

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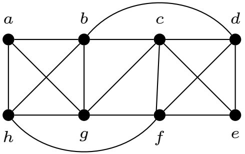
$G =$ 

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The larger the κ, the more connected the graph. We say G is k-connected if $\kappa(G) \geqslant k$.

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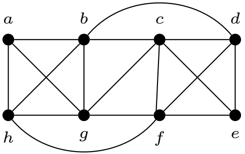
$G =$ 

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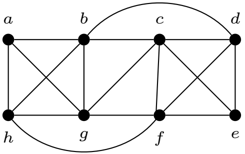

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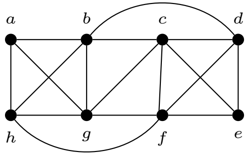

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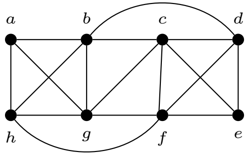
$G =$ 

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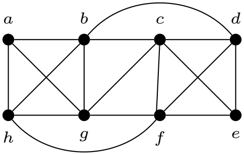
$G =$ 

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$$\kappa(G) \leqslant \lambda(G) \leqslant \min_{v \in V}(\deg(v)).$$

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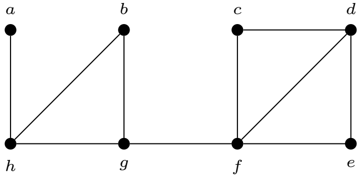

$\kappa(G) :$

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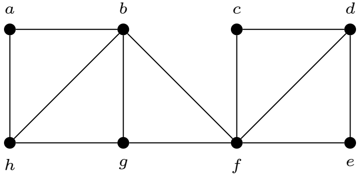
$\min_{v \in V}(\deg(v)) :$

$H =$



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Graph invariants

Recall, a graph invariant is a statistic about a graph that is preserved under isomorphisms (relabeling of the vertices). Namely, if you don't need the labels to calculate the statistic, then it's probably a graph invariant.

1. $|V|$, $|E|$
2. Degree sequence

   Also: Minimum degree, maximum degree, vertex of degree d_1 adjacent to vertex of degree d_2, ...
3. Bipartite or not

   If any subgraph is not bipartite, then G is not bipartite. A graph is bipartite if and only if it has no odd cycles as subgraphs.
4. Paths or cycles of particular lengths

   Also: longest path or cycle length, maximal paths of certain lengths, ...
5. Edge and vertex connectivity

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You try: Exercise 52.

Aside: necessary and sufficient conditions

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Some examples:

- Event: Passing 365; NC: take all three exams.
- Event: Staying alive; NC: breathing.
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A sufficient condition is a condition or set of conditions that will produce the event.
Some examples:

- Event: Passing 365;
  SC: do all of the homework and get 100% on all exams and quizzes.
- Event: Being a parent; SC: having a daughter.
- Event: $x^2 = 1$; SC: $x = -1$.

Sufficient conditions	imply	Event	implies	Necessary conditions

Eulerian trails and circuits

Suppose you're trying to design a maximally efficient route for postal delivery, or street cleaning. You want walk on the city streets that visits every street exactly once.

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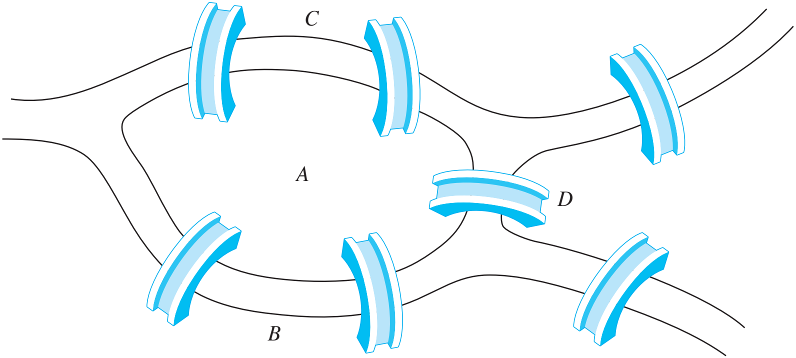
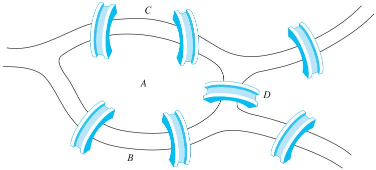
"The Seven Bridges of Königsberg", Leonhard Euler (1736)

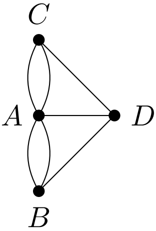


Question: is it possible to start at some location in the town, travel across all the bridges once without crossing any bridge twice?

Eulerian trails and circuits

What Euler did was model the problem as the multigraph





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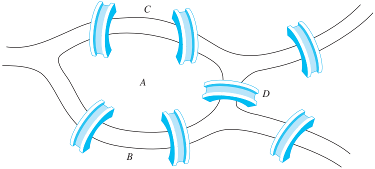

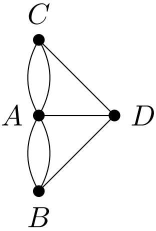

In honor of his contribution, we say that an Eulerian trail in a graph G is a trail (no repeated edges) that passes through every edge of G (exactly once).

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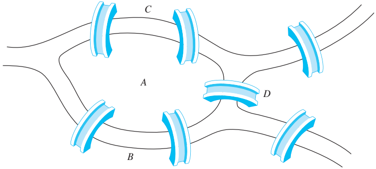

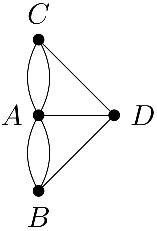

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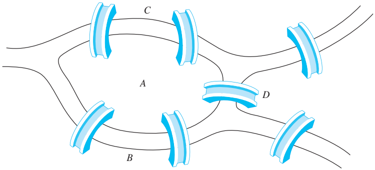

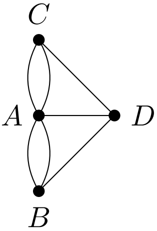

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Necessary: Connected

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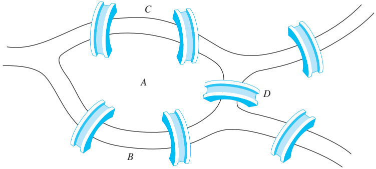

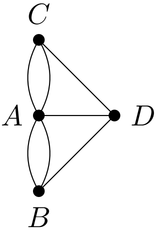

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Necessary: Connected; at most two vertices of odd degree.

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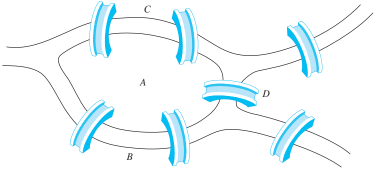

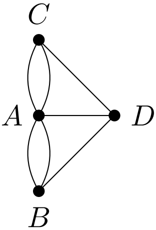

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Algorithm for finding an Eulerian circuit in any graph with all even degree vertices: Start anywhere and go until you get stuck – you'll be back where you started. Somewhere in the middle, you have a vertex where you didn't exhaust the edges incident. Go back and start from there and go until you get stuck. Repeat.

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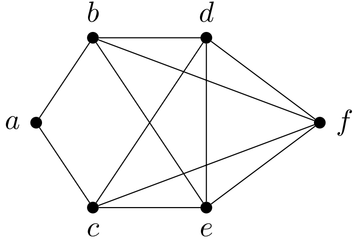

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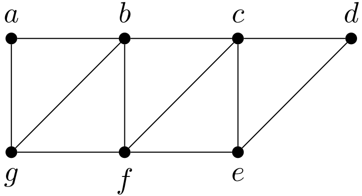
Algorithm for finding an Eulerian trail in any graph with all but two even degree vertices: Start at an odd-degree vertex and go until you get stuck. Somewhere in the middle, you have a vertex where you didn't exhaust the edges incident. Go back and start from there and go until you get stuck. Repeat.



Eulerian trails and circuits

Theorem

A graph has an Eulerian trail if and only if it is connected and has at most two vertices of odd degree. Further, a connected graph has an Eulerian circuit if and only if every vertex is of even degree.

Hamilton paths and cycles

A Hamilton path is a path in G that visits every vertex exactly once. A Hamilton cycle is a cycle that visits every vertex in G.

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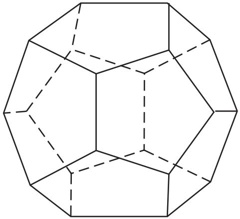
Famous puzzle: start with a dodecahedron, and imagine the vertices are cities around the world, and the edges are routes from one city to the next. The goal is to visit every city exactly once.



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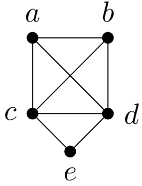

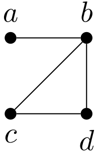

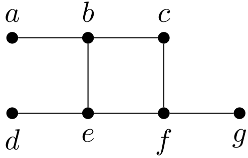

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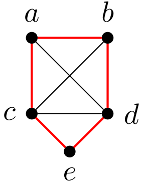

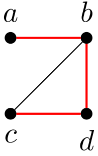

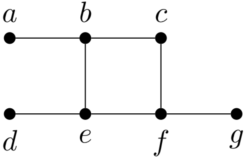

The first has a Hamilton cycle, the second has a Hamilton path, the third has neither (you'd get stuck at a, d, or g without hitting at least one of those three).

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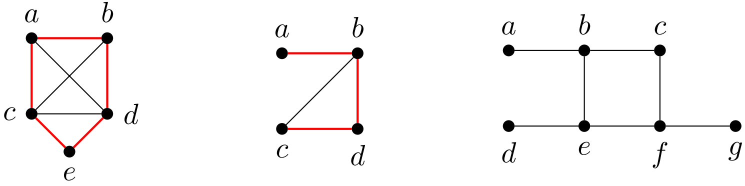

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See also: traveling salesman problem
"What is the shortest route a traveling salesperson should take to visit a set of cities?"

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In contrast to Eulerian trails, **there are no simple necessary and sufficient conditions for the existence of Hamilton paths and cycles**.

Necessary conditions

- Paths: no more than two vertices of degree 1.

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  - If a vertex has degree 2, you know both edges incident must be in the cycle.
  - No cut vertices or edges.

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Sufficient conditions

Note: the more edges a graph has, the more likely it is that there's a Hamilton cycle.

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If G is a simple connected graph with $n \geqslant 3$ vertices, such that

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then G has a Hamilton circuit.

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Ore's Theorem

If G is a simple connected graph with $n \geqslant 3$ vertices such that

$$\deg(u) + \deg(v) \geqslant n$$

for every pair of non-adjacent vertices u and v, then G has a Hamilton circuit.

(Note that Ore's theorem implies Dirac's theorem.)