Math 365 – Monday 4/1/19

8.5 (inclusion/exclusion) and 9.1 & 9.5 (equivalence relations)

Exercise 41. Use inclusion/exclusion to answer the following questions.

(a) How many elements are in $A_1 \cup A_2$ if there are 12 elements in A_1, 18 elements in A_2, and
  (i) $|A_1 \cap A_2| = 6$?
  (ii) $A_1 \cap A_2 = \emptyset$?
  (iii) $A_1 \subseteq A_2$?

(b) A survey of households in the United States reveals that 96% have at least one television set, 42% have a land-line telephone service, and 39% have land-line telephone service and at least one television set. What percentage of households in the United States have neither telephone service nor a television set?
[Start by naming your sets, as in "Let A be the set of households that have at least one TV set," and so on.]

(c) How many students are enrolled in a course either in
(1) *calculus 1*, (2) *discrete math*,
(3) *data structures*, or (4) *intro to computer science*
at a school if there are 507, 292, 312, and 344 students in these courses, respectively; 14 in both calculus and data structures; 213 in both calculus and intro to CS; 211 in both discrete mathematics and data structures; 43 in both discrete mathematics and intro to CS; and no student may take calculus and discrete mathematics at the same time, nor intro to CS and data structures at the same time?
[Again, start by naming your sets, as in "Let A be the set of students enrolled in calculus 1," and so on.]

(d) Find the number of integers $1 \leq n \leq 100$ that are odd and/or the square of an integer.

(e) Find the number of integers $1 \leq n \leq 500$ that are *not* a multiple of 3, 5, or 7.

Exercise 42. Recall that the *Stirling numbers (of the second kind)* count arrangements of distinguishable objects into indistinguishable boxes, namely

$$S(n,k) = \left| \left\{ \begin{array}{c} \text{Ways to place } n \text{ distinguishable objects} \\ \text{into } k \text{ indistinguishable boxes} \\ \text{so that no box is left empty} \end{array} \right\} \right|.$$

We stated in section 6.5 that

$$S(n,k) = \frac{1}{k!} \sum_{\ell=0}^{k-1} (-1)^\ell \binom{n}{\ell} (k-\ell)^n.$$

We can now check this using inclusion/exclusion!

But first, we count the number of surjective functions from $X = \{1, 2, \ldots, n\}$ to $Y = \{1, \ldots, k\}$ (where $k \leq n$). To that end, let U be the set of *all* functions from X to Y, and for $i = 1, \ldots, k$, let

$$A_i = \{\text{functions } f : X \to Y \mid i \notin f(X)\}.$$

(a) What is $|U|$, i.e. how many functions are there from X to Y?
[Don't put any restrictions on the functions here—this is a simple product rule question.]

(b) For $X = \{1, 2, 3\}$ and $Y = \{1, 2, 3\}$, we have

$$A_1 = \{f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8\}$$

where

$$f_1 \text{ sends } \begin{array}{r} 1 \mapsto 2, \\ 2 \mapsto 2, \\ 3 \mapsto 2; \end{array} \quad f_2 \text{ sends } \begin{array}{r} 1 \mapsto 3, \\ 2 \mapsto 2, \\ 3 \mapsto 2; \end{array} \quad f_3 \text{ sends } \begin{array}{r} 1 \mapsto 2, \\ 2 \mapsto 3, \\ 3 \mapsto 2; \end{array} \quad f_4 \text{ sends } \begin{array}{r} 1 \mapsto 2, \\ 2 \mapsto 2, \\ 3 \mapsto 3; \end{array}$$

$$f_5 \text{ sends } \begin{array}{r} 1 \mapsto 3, \\ 2 \mapsto 3, \\ 3 \mapsto 2; \end{array} \quad f_6 \text{ sends } \begin{array}{r} 1 \mapsto 3, \\ 2 \mapsto 2, \\ 3 \mapsto 3; \end{array} \quad f_7 \text{ sends } \begin{array}{r} 1 \mapsto 2, \\ 2 \mapsto 3, \\ 3 \mapsto 3; \end{array} \quad \text{and} \quad f_8 \text{ sends } \begin{array}{r} 1 \mapsto 3, \\ 2 \mapsto 3, \\ 3 \mapsto 3. \end{array}$$

(i) What is A_2? [Describe the *set*, not its size.]
(ii) What is $A_1 \cap A_2$?
(iii) What is $A_1 \cap A_2 \cap A_3$?
[You should be able to do this without computing A_3.]

(c) Explain why, for general n and k, we have the following:
(i) $|A_1| = (k-1)^n$;
(ii) $|A_1 \cap A_2| = (k-2)^n$;
(iii) $|A_1 \cap A_2 \cap \cdots \cap A_\ell| = (k-\ell)^n$ (for any $\ell \le k$);
(iv) $|A_1 \cap A_2 \cap \cdots \cap A_k| = 0$.

(d) Explain why for any subset $S \subseteq \{A_1, A_2, \ldots, A_k\}$ of size ℓ, we have

$$\left| \bigcap_{A_i \in S} A_i \right| = |A_1 \cap \cdots \cap A_\ell|.$$

[For example $|A_1 \cap A_3 \cap A_7| = |A_1 \cap A_2 \cap A_3|$.]

(e) Use inclusion/exclusion to give a formula for $|A_1 \cup A_2 \cup \cdots \cup A_k|$.

(f) Explain why the set of surjective functions $f : X \to Y$ is

$$\overline{A_1 \cup A_2 \cup \cdots \cup A_k}, \quad \text{i.e.} \quad U - A_1 \cup A_2 \cup \cdots \cup A_k.$$

(g) Use the last two parts, together with part (a), to give the number of surjective functions from X to Y. [Your answer should line up Theorem 1 in Section 8.6.]

(h) Use division rule to explain why $S(n, k)$ is $\frac{1}{k!}$ times your answer to (g). Check that this agrees with the formula we gave above.

Exercise 43. (Relations)

(a) Which of these relations on $\{0, 1, 2, 3\}$ are equivalence relations? For those that are not, what properties do they lack?

(i) $\{0 \sim 0, 1 \sim 1, 2 \sim 2, 3 \sim 3\}$
(ii) $\{0 \sim 0, 0 \sim 2, 2 \sim 0, 2 \sim 2, 2 \sim 3, 3 \sim 2, 3 \sim 3\}$
(iii) $\{0 \sim 0, 1 \sim 1, 1 \sim 2, 2 \sim 1, 2 \sim 2, 3 \sim 3\}$
(iv) $\{0 \sim 0, 1 \sim 1, 1 \sim 3, 2 \sim 2, 2 \sim 3, 3 \sim 1, 3 \sim 2, 3 \sim 3\}$
(v) $\{0 \sim 0, 0 \sim 1, 0 \sim 2, 1 \sim 0, 1 \sim 1, 1 \sim 2, 2 \sim 0, 2 \sim 2, 3 \sim 3\}$

(b) For each of the equivalence relations in part (a), list the equivalence classes.

(c) Which of these relations on the set of all people are equivalence relations? For those that are not, what properties do they lack?

(i) $a \sim b$ if a and b are the same age;
(ii) $a \sim b$ if a and b have the same parents;
(iii) $a \sim b$ if a and b share a common parent;
(iv) $a \sim b$ if a and b have met;
(v) $a \sim b$ if a and b speak a common language.

(d) For the following relations on A determine whether they are reflexive, symmetric, and/or transitive. State whether they are equivalence relations or not, and if they are describe their equivalence classes.

(a) Let $A = \mathbb{Z}$ and define $\sim$ by $a \sim b$ whenever $a - b$ is odd.
(b) Let $A = \mathbb{R}$ and define $\sim$ by $a \sim b$ whenever $ab \neq 0$.
(c) Let $A = \{f : \mathbb{Z} \to \mathbb{Z}\}$ and define $\sim$ by $f \sim g$ whenever $f(1) = g(1)$.

(e) Verify that the relation

$$f(x) \sim g(x) \qquad \text{if} \qquad \frac{d}{dx} f(x) = \frac{d}{dx} g(x)$$

is an equivalence relation on the set

$$D = \{\text{differentiable functions } \varphi : \mathbb{R} \to \mathbb{R}\},$$

and describe the set of functions that are equivalent to $f(x) = x^2$.

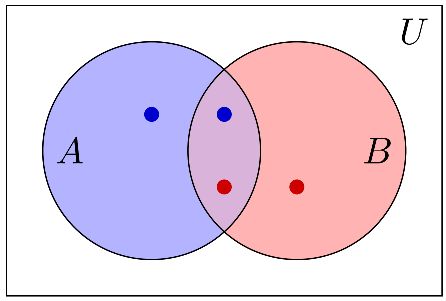
Inclusion/exclusion

Recall the "subtraction" rule:

For two sets A and B, we have

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

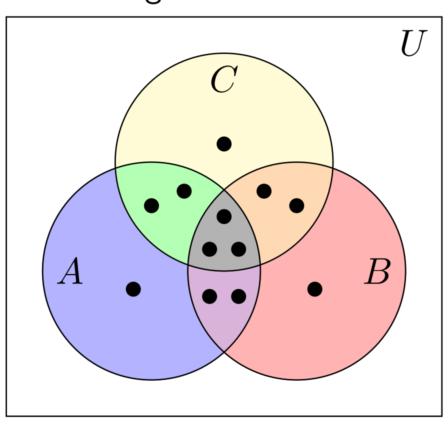
Venn diagram for $A \cup B$:



Inclusion/exclusion

For three sets A, B, and C...

Venn diagram for $A \cup B \cup C$:



$$\begin{aligned} |A \cup B \cup C| = |A| + |B| + |C| & \quad \text{"include"} \\ - (|A \cap B| + |A \cap C| + |B \cap C|) & \quad \text{"exclude"} \\ + |A \cap B \cap C|. & \quad \text{"include"} \end{aligned}$$

Example: How many integers are there $1 \leqslant n \leqslant 100$ that are multiples of 2, 3, and/or 5?

Ans. Let $U = \{n \in \mathbb{Z} \mid 1 \leqslant n \leqslant 100\}$,

$$A = \{n \in U \mid n \text{ is a multiple of } 2\},$$
$$B = \{n \in U \mid n \text{ is a multiple of } 3\}, \quad \text{and}$$
$$C = \{n \in U \mid n \text{ is a multiple of } 5\};$$

so we want to know the size of

$$A \cup B \cup C = \{n \in U \mid n \text{ is a multiple of } 2,\ 3, \text{ and/or } 5\}.$$

To use inclusion/exclusion, we need to compute the following:

$|A| = \lfloor 100/2 \rfloor = 50$ $\qquad$ $|A \cap B| = \lfloor 100/(2*3) \rfloor = 16$

$|B| = \lfloor 100/3 \rfloor = 33$ $\qquad$ $|A \cap C| = \lfloor 100/(2*5) \rfloor = 10$

$|C| = \lfloor 100/5 \rfloor = 20$ $\qquad$ $|B \cap C| = \lfloor 100/(3*5) \rfloor = 6$

$$|A \cap B \cap C| = \lfloor 100/(2*3*5) \rfloor = 3$$

So

$$|A \cup B \cup C| = 50 + 33 + 20 - 16 - 10 - 6 + 3 = \boxed{74}.$$

Inclusion/exclusion

Thm. For sets A_1, A_2, ..., A_n, we have

$$|A_1 \cup \cdots \cup A_n| = \sum_{S \subseteq \{A_1,\ldots,A_n\}} (-1)^{|S|-1} \left| \bigcap_{A_i \in S} A_i \right|$$

Process this statement for $n = 3$:

Start with sets A_1, A_2, and A_3...

Try Exercises 41 & 42

S	$\varnothing$	$\{A_1\}$	$\{A_2\}$	$\{A_3\}$
$\lvert S\rvert$	0	1	1	1
$(-1)^{\lvert S\rvert-1}$	-1	1	1	1
$\bigcap_{A_i \in S} A_i$	$\varnothing$	A_1	A_2	A_3

S	$\{A_1, A_2\}$	$\{A_1, A_3\}$	$\{A_2, A_3\}$	$\{A_1, A_2, A_3\}$
$\lvert S\rvert$	2	2	2	3
$(-1)^{\lvert S\rvert-1}$	-1	-1	-1	1
$\bigcap_{A_i \in S} A_i$	$A_1 \cap A_2$	$A_1 \cap A_3$	$A_2 \cap A_3$	$A_1 \cap A_2 \cap A_3$

Relations

A binary relation on a set A is a subset $R \subseteq A \times A$, where elements (a, b) are written as $a \sim b$.

Example: $A = \mathbb{Z}$ and $R = \{a \sim b \mid a < b\}$.
In words:

Let $\sim$ be the relation on $\mathbb{Z}$ given by $a \sim b$ whenever $a < b$.

Example: $A = \mathbb{R}$ and $R = \{a \sim b \mid a = b\}$.
In words:

Let $\sim$ be the relation on $\mathbb{R}$ given by $a \sim b$ whenever $a = b$.

Example: $A = \mathbb{Z}$ and

$R = \{a \sim b \mid a$ and b have the same remainder when divided by 3$\}$.

More examples of (binary) relations:

1. For A a number system, let $a \sim b$ if $a = b$. R, S, T
2. For A a number system, let $a \sim b$ if $a < b$. not R, not S, T
3. For $A = \mathbb{R}$, let $a \sim b$ if $ab = 0$. not R, S, not T
4. For A a set of people, let $a \sim b$ if a is a (full) sibling of b. not R, S, T
5. For A a set of people, let $a \sim b$ if a and b speak a common language. R, S, not T

A binary relation on a set A is...

(R) reflexive if $a \sim a$ for all $a \in A$;

(S) symmetric if $a \sim b$ implies $b \sim a$;

(T) transitive if $a \sim b$ and $b \sim c$ implies $a \sim c$, i.e.

$$(a \sim b \wedge b \sim c) \Rightarrow a \sim c$$

An equivalence relation on a set A is a binary relation that is reflexive, symmetric, *and* transitive. (Only #1)

Fix $n \in \mathbb{Z}_{>0}$ and define the relation on $\mathbb{Z}$ given by

"$a \sim b$ whenever a and b have the same remainder when divided by n."

Is $\sim$ is an equivalence relation?

Note: Having the same remainder means that

$a - b$ is a multiple of n.

For example, let $n = 5$:

integer:	-3	-2	-1	0	1	2	3	4	5	6	7	8
remainder:	2	3	4	0	1	2	3	4	0	1	2	3

So $0 \sim 5$, and $-2 \sim 3 \sim 8$, but $-3 \not\sim 3$.

Check: we have $a \sim b$ whenever $a - b = kn$ for some $k \in \mathbb{Z}$.

reflexivity: $a - a = 0 = 0 \cdot n$ ✓

symmetry: If $a - b = kn$, then $b - a = -kn = (-k)n$. ✓

transitivity: If $a - b = kn$ and $b - c = \ell n$, then

$$a - c = (a - b) + (b - c) = kn + \ell n = (k + \ell)n. ✓$$

Yes! This is an equivalence relation!

Let A be a set. Consider the relation on $\mathcal{P}(A)$ by

$$S \sim T \qquad \text{if} \qquad S \subseteq T$$

Is $\sim$ is an equivalence relation?

Check: This is reflexive and transitive, but not symmetric.
So no, it is not an equivalence relation.

Is

$$S \sim T \qquad \text{if} \qquad S \subseteq T \text{ or } S \subseteq T$$

an equivalence relation on $\mathcal{P}(A)$?

Check: This is reflexive and symmetric, but not transitive.
So still no, it is not an equivalence relation.

Is

$$S \sim T \qquad \text{if} \qquad |S| = |T|$$

an equivalence relation on $\mathcal{P}(A)$?

Let $\sim$ be an equivalence relation on a set A, and let $a \in A$. The set of all elements $b \in A$ such that $a \sim b$ is called the equivalence class of a, denoted by $[a]$.

Example: Consider the equivalence relation on $A = \{a, b, c\}$ given by

$$a \sim a, \quad b \sim b, \quad c \sim c, \quad a \sim c, \quad \text{and} \quad c \sim a.$$

Then

$$[a] = \{a, c\} = [c], \quad \text{and}$$
$$[b] = \{b\}$$

are the two equivalence classes in A (with respect to this relation).

(We say there are two, *not three*, since "the equivalence classes" refers to the sets themselves, not to the elements that generate them.)

Let $\sim$ be an equivalence relation on a set A, and let $a \in A$. The set of all elements $b \in A$ such that $a \sim b$ is called the equivalence class of a, denoted by $[a]$.

Example: We showed that

$$\text{“}a \sim b \quad \text{if } a - b = 5k \text{ for some } k \in \mathbb{Z}\text{”}$$

is an equivalence relation on $\mathbb{Z}$. Then

$$[0] = \{5n \mid n \in \mathbb{Z}\} = 5\mathbb{Z} \qquad [1] = \{5n + 1 \mid n \in \mathbb{Z}\} = 5\mathbb{Z} + 1$$
$$[2] = \{5n + 2 \mid n \in \mathbb{Z}\} = 5\mathbb{Z} + 2 \qquad [3] = \{5n + 3 \mid n \in \mathbb{Z}\} = 5\mathbb{Z} + 3$$
$$[4] = \{5n + 4 \mid n \in \mathbb{Z}\} = 5\mathbb{Z} + 4$$
$$[5] = \{5n + 5 \mid n \in \mathbb{Z}\} = \{5m \mid m \in \mathbb{Z}\} = [0] = [-5] = [10] = \cdots$$
$$[6] = \{5n + 6 \mid n \in \mathbb{Z}\} = \{5m + 1 \mid m \in \mathbb{Z}\} = [1] = [-4] = [11] = \cdots$$
$$\vdots$$

Theorem. The equivalence classes of A partition A into subsets, meaning

1. the equivalence classes are subsets of A:
$$[a] \subseteq A \text{ for all } a \in A;$$
2. any two equivalence classes are either equal or disjoint:
for all $a, b \in A$, either $[a] = [b]$ or $[a] \cap [b] = \varnothing$; and
3. the union of all the equivalence classes is all of A:
$$A = \bigcup_{a \in A} [a].$$

We say that A is the disjoint union of equivalency classes, written
$$A = \bigsqcup_{a \in A} [a], \qquad \text{LaTeX: \textbackslash bigsqcup, \textbackslash sqcup}$$

For example, in our last example, there are exactly 5 equivalence classes: $[0]$, $[1]$, $[2]$, $[3]$, and $[4]$. Any other seemingly different class is actually one of these (for example, $[5] = [0]$). And
$$[0] \cup [1] \cup [2] \cup [3] \cup [4] = \mathbb{Z}.$$

So $\boxed{\mathbb{Z} = [0] \sqcup [1] \sqcup [2] \sqcup [3] \sqcup [4]}$.