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A linear homogeneous recurrence relation of degree $k$ with
constant coefficients is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k},$$

where $c_1, c_2, \ldots, c_k$ are real numbers, and $c_k \neq 0$. 
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where $c_1, c_2, \ldots, c_k$ are real numbers, and $c_k \neq 0$.

Solutions to this kind of relation come in the form $r^n$, where $r$ is a root of the characteristic equation, which is obtained by plugging in $r^n$, dividing through by $r^{n-k}$, and solving for 0:

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_{k-1} r - c_k = 0.$$

The roots of this equation are called the characteristic roots.
Example from last time: plugging $a_n = r^n$ into the recursion relation

$$a_n = -a_{n-1} + a_{n-2} + a_{n-3}$$

gets

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$$0 = 0 = (r + 1)^2(r - 1).$$

characteristic equation
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\]

which is true if and only if \( r = 0 \) or

\[
\underbrace{0 = r^3 + r^2 - r - 1}_{\text{characteristic equation}} = (r + 1)^2(r - 1).
\]

The characteristic roots are \( r_1 = 1 \) (with multiplicity 1) and \( r_2 = -1 \) (with multiplicity 2).
Theorem: Solving linear homogeneous recurrences

Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose that the characteristic equation has roots $r_1, r_2, \ldots, r_\ell$ with multiplicities $m_1, m_2, \ldots, m_\ell$. Then a sequence $\{a_n\}_{n \in \mathbb{N}}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$ if and only if

$$a_n = p_1(n) r_1^n + p_2(n) r_2^n + \cdots + p_\ell(n) r_\ell^n,$$

where $p_i(n)$ are polynomials in $n$ of degree $m_i - 1$. 
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Example: $a_n = a_{n-1} + 5a_{n-2} - a_{n-3} - 8a_{n-4} - 4a_{n-5}$. 
Theorem: Solving linear homogeneous recurrences
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Characteristic equation:
\[
0 = r^5 - r^4 - 5r^3 + r^2 + 8r + 4
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Example: \( a_n = a_{n-1} + 5a_{n-2} - a_{n-3} - 8a_{n-4} - 4a_{n-5} \).
Characteristic equation:
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0 = r^5 - r^4 - 5r^3 + r^2 + 8r + 4 = (r + 1)^3(r - 2)^2.
\]
General solution:
\[
a_n = (\alpha_0 + \alpha_1n + \alpha_2n^2)(-1)^n + (\beta_0 + \beta_1n)(2)^n.
\]
Non-homogeneous equations

Suppose your recurrence is linear and constant coefficient in $a_i$’s, but is not homogeneous. In other words, it is in the form

\[ a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n) \]

where $F(n)$ is a function only in $n$ (no $a_i$’s).
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$$h_n = c_1 h_{n-1} + c_2 h_{n-2} + \cdots + c_k h_{n-k} \quad \text{(so that } a_n = h_n + F(n)\text{)}$$

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Ex: \(a_n = 3a_{n-1} + 2n\).
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**Ex:** \( a_n = 3a_{n-1} + 2n \).

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F(n) = 2n, \quad \text{Assoc. hom: } h_n = 3h_{n-1}.
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Ex: $a_n = 5a_{n-1} - 6a_{n-2} + 7^n.$
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The following theorem says that if we can find **one solution** to \( a_n \), then the general solutions to \( h_n \) will help us build all the rest of the solutions to \( a_n \).
Theorem: Solving non-homogeneous equations

(a) If \( \{\hat{a}_n\}_{n \in \mathbb{N}} \) is one solution of the non-homogeneous linear recurrence relation with constant coefficients
\[
a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n),
\]
then every solution is of the form \( \{a_n = \hat{a}_n + h_n\}_{n \in \mathbb{N}} \), where \( \{h_n\}_{n \in \mathbb{N}} \) is a solution of the associated homogeneous recurrence relation.

(b) Finding \( \hat{a}_n \): If \( F(n) = Q(n)R^n \), where
\[
\begin{align*}
&\quad Q(n) \text{ is a polynomial in } n, \text{ and} \\
&\quad R \text{ is a constant,}
\end{align*}
\]
then there is a solution to \( a_n \) of the form
\[
\hat{a}_n = n^m q(n) R^n
\]
where
\[
\begin{align*}
&\quad \deg(q(n)) \leq \deg(Q(n)), \text{ and} \\
&\quad m = \text{mult. of } R \text{ in the characteristic equation (possibly 0).}
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Ex: Find all solutions to \( a_n = 3a_{n-1} + 2n \). What is the solution with \( a_1 = 3 \)?
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Ex: Find all solutions to \( a_n = 3a_{n-1} + 2n \). What is the solution with \( a_1 = 3 \)?
1. Break the sequence into two parts: homogeneous $h_n$ and a function of $n$: $a_n = h_n + F(n)$.

2. Solve for $h_n$:
   - compute the characteristic equation;
   - factor to compute roots and multiplicities;
   - build the general solution to $h_n$.

3. Find one solution $\hat{a}_n$ by guessing something of a similar form.

   If $F(n) = Q(n)R^n$, guess $\hat{a}_n = n^m q(n) R^n$

   where $m = \text{mult of } R$, and $q_n = b_0 + b_1n + b_2n^2 + \cdots + b_d n^d$

   where $d = \deg(Q(n))$. 

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Example: $a_n = 4a_{n-1} + 3a_{n-2} - 18a_{n-3} + 4^n$
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Homog: $h_n = 4h_{n-1} + 3h_{n-2} - 18h_{n-3}$ and $F(n) = 4^n$. 
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Char eq: $0 = r^3 - 4r^2 - 3r + 18$
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Char eq: $0 = r^3 - 4r^2 - 3r + 18 = (r + 2)(r - 3)^2$
1. Break the sequence into two parts: homogeneous $h_n$ and a function of $n$: 
$$a_n = h_n + F(n).$$

2. Solve for $h_n$:
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   If $F(n) = Q(n)R^n$, guess $\hat{a}_n = n^m q(n)R^n$

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$$\text{Homog: } h_n = 4h_{n-1} + 3h_{n-2} - 18h_{n-3} \quad \text{and} \quad F(n) = 4^n.$$  

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Particular solution guess: $\hat{a}_n = b4^n$
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Homog: $h_n = 4h_{n-1} + 3h_{n-2} - 18h_{n-3}$ and $F(n) = 4^n$.
Char eq: $0 = r^3 - 4r^2 - 3r + 18 = (r + 2)(r - 3)^2$
Homog sol: $h_n = \alpha(-2)^n + (\alpha + \beta n)3^n$
Particular solution guess: $\hat{a}_n = b4^n$ (gives $b = 32/3$)
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(gives $b_0 = 1376/9$ and $b_1 = -63/3$)
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General sol: $a_n = \alpha(-2)^n + (\alpha + \beta n)3^n + \left(\frac{1376}{9} - \frac{63}{3}n\right)4^n$. 
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Example: $a_n = 4a_{n-1} + 3a_{n-2} - 18a_{n-3} + n3^n$

Homog: $h_n = 4h_{n-1} + 3h_{n-2} - 18h_{n-3}$ and $F(n) = 3^n$.

Char eq: $0 = r^3 - 4r^2 - 3r + 18 = (r + 2)(r - 3)^2$

Homog sol: $h_n = \alpha(-2)^n + (\alpha + \beta n)3^n$

Particular solution guess: $\hat{a}_n = (b_0 + b_1n)n^23^n$

(gives $b_0 = 21/50$ and $b_1 = 1/10$)
1. Break the sequence into two parts: homogeneous $h_n$ and a function of $n$: $a_n = h_n + F(n)$.

2. Solve for $h_n$:
   - compute the characteristic equation;
   - factor to compute roots and multiplicities;
   - build the general solution to $h_n$.

3. Fine one solution $\hat{a}_n$ by guessing something of a similar form.
   
   If $F(n) = Q(n)R^n$, guess $\hat{a}_n = n^m q(n) R^n$
   
   where $m = \text{mult of } R$, and $q_n = b_0 + b_1 n + b_2 n^2 + \cdots + b_d n^d$
   
   where $d = \deg(Q(n))$.

Example: $a_n = 4a_{n-1} + 3a_{n-2} - 18a_{n-3} + n3^n$

Homog: $h_n = 4h_{n-1} + 3h_{n-2} - 18h_{n-3}$ and $F(n) = 3^n$.

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Particular solution guess: $\hat{a}_n = (b_0 + b_1 n) n^2 3^n$

(gives $b_0 = 21/50$ and $b_1 = 1/10$)

General sol: $a_n = \alpha(-2)^n + (\alpha + \beta n)3^n + \left(\frac{21}{50} + \frac{1}{10} n\right)n^2 3^n$. 
Taylor series to know and love:

\[ p \frac{1}{1-x^n} = \sum_{k=0}^{\infty} x^k \]

Combining series: Let \( f_p(x^{1/q}) \) and \( g_p(x^{1/q}) \). Then

\[ f_p(x^{1/q}) + g_p(x^{1/q}) = \sum_{k=0}^{\infty} a_k x^k + \sum_{k=0}^{\infty} b_k x^k = \sum_{k=0}^{\infty} (a_k + b_k) x^k \]

\[ \sum_{k=0}^{\infty} \left( \sum_{i=0}^{k} a_i b_{k-i} \right) x^k \]
Section 8.4: Generating functions.

Taylor series to know and love:

\[(1 + x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k = 1 + \binom{n}{1} x + \binom{n}{2} x^2 + \cdots + x^n \quad \text{(finite)}\]

\[\frac{1 - x^n}{1 - x} = \sum_{k=0}^{n-1} x^k = 1 + x + x^2 + \cdots + x^{n-1} \quad \text{(finite)}\]

\[\frac{1}{1 - x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots \quad \text{(infinite)}\]

\[e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots \quad \text{(infinite)}\]
Section 8.4: Generating functions.

Taylor series to know and love:

\[(1 + x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k = 1 + \binom{n}{1} x + \binom{n}{2} x^2 + \cdots + x^n \] (finite)

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\[e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots \] (infinite)

Combining series: Let \( f(x) = \sum_{k=0}^{\infty} a_k x^k \) and \( g(x) = \sum_{k=0}^{\infty} b_k x^k \). Then

\[ f(x) + g(x) = \sum_{k=0}^{\infty} (a_k + b_k) x^k \] and

\[ f(x) g(x) = \sum_{k=0}^{\infty} \left( \sum_{i=0}^{k} a_i b_{k-i} \right) x^k. \]
Example: Compute the series for \( \frac{1}{(1-x)^2} \) using \( \frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \).
Example: Compute the series for \( \frac{1}{(1-x)^2} \) using \( \frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \).

**Approach 1:** Use the multiplication rule,

\[
\left( \sum_{k=0}^{\infty} a_k x^k \right) \left( \sum_{k=0}^{\infty} b_k x^k \right) = \sum_{k=0}^{\infty} \left( \sum_{i=0}^{k} a_i b_{k-i} \right) x^k,
\]

on

\[
\frac{1}{(1-x)^2} = \frac{1}{1-x} \cdot \frac{1}{1-x} = \left( \sum_{k=0}^{\infty} x^k \right) \left( \sum_{k=0}^{\infty} x^k \right).
\]
Example: Compute the series for $\frac{1}{(1-x)^2}$ using $\frac{1}{1-x} = \sum_{k=0}^\infty x^k$.

Approach 1: Use the multiplication rule,

$$\left(\sum_{k=0}^\infty a_kx^k\right)\left(\sum_{k=0}^\infty b_kx^k\right) = \sum_{k=0}^\infty \left(\sum_{i=0}^k a_ib_{k-i}\right)x^k,$$

on

$$\frac{1}{(1-x)^2} = \frac{1}{1-x} \ast \frac{1}{1-x} = \left(\sum_{k=0}^\infty x^k\right)\left(\sum_{k=0}^\infty x^k\right).$$

Here, $a_i = b_i = 1$ for all $i$. 
Example: Compute the series for $\frac{1}{(1-x)^2}$ using $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$.

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$$\left( \sum_{k=0}^{\infty} a_k x^k \right) \left( \sum_{k=0}^{\infty} b_k x^k \right) = \sum_{k=0}^{\infty} \left( \sum_{i=0}^{k} a_i b_{k-i} \right) x^k,$$

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So

$$\sum_{i=0}^{k} a_i b_{k-i}$$
Example: Compute the series for \( \frac{1}{(1-x)^2} \) using \( \frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \).

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\frac{1}{(1-x)^2} = \frac{1}{1-x} \ast \frac{1}{1-x} = \left( \sum_{k=0}^{\infty} x^k \right) \left( \sum_{k=0}^{\infty} x^k \right).
\]

Here, \( a_i = b_i = 1 \) for all \( i \).

So

\[
\sum_{i=0}^{k} a_i b_{k-i} = \sum_{i=0}^{k} 1 \ast 1
\]
Example: Compute the series for \( \frac{1}{(1-x)^2} \) using \( \frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \).

Approach 1: Use the multiplication rule,

\[
\left( \sum_{k=0}^{\infty} a_k x^k \right) \left( \sum_{k=0}^{\infty} b_k x^k \right) = \sum_{k=0}^{\infty} \left( \sum_{i=0}^{k} a_i b_{k-i} \right) x^k,
\]

on

\[
\frac{1}{(1-x)^2} = \frac{1}{1-x} \cdot \frac{1}{1-x} = \left( \sum_{k=0}^{\infty} x^k \right) \left( \sum_{k=0}^{\infty} x^k \right).
\]

Here, \( a_i = b_i = 1 \) for all \( i \).

So

\[
\sum_{i=0}^{k} a_i b_{k-i} = \sum_{i=0}^{k} 1 \cdot 1 = k + 1.
\]
Example: Compute the series for $\frac{1}{(1-x)^2}$ using $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$.

Approach 1: Use the multiplication rule,

$$\left( \sum_{k=0}^{\infty} a_k x^k \right) \left( \sum_{k=0}^{\infty} b_k x^k \right) = \sum_{k=0}^{\infty} \left( \sum_{i=0}^{k} a_i b_{k-i} \right) x^k,$$

on

$$\frac{1}{(1-x)^2} = \frac{1}{1-x} * \frac{1}{1-x} = \left( \sum_{k=0}^{\infty} x^k \right) \left( \sum_{k=0}^{\infty} x^k \right).$$

Here, $a_i = b_i = 1$ for all $i$.

So

$$\sum_{i=0}^{k} a_i b_{k-i} = \sum_{i=0}^{k} 1 * 1 = k + 1.$$

Thus

$$\frac{1}{(1-x)^2} = \sum_{k=0}^{\infty} (k + 1)x^k.$$
Example: Compute the series for $\frac{1}{(1-x)^2}$ using $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$.

Approach 1: Use the multiplication rule,

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Here, $a_i = b_i = 1$ for all $i$.

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$$\sum_{i=0}^{k} a_i b_{k-i} = \sum_{i=0}^{k} 1 \cdot 1 = k + 1.$$

Thus

$$\frac{1}{(1-x)^2} = \sum_{k=0}^{\infty} (k + 1) x^k = 1 + 2x + 3x^2 + 4x^3 + \cdots.$$
Example: Compute the series for $\frac{1}{(1-x)^2}$ using $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$. 
**Example:** Compute the series for \( \frac{1}{(1-x)^2} \) using \( \frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \).

**Approach 2:** Use derivatives, noting that

\[
\frac{d}{dx} \frac{1}{1-x} = \frac{d}{dx} (1-x)^{-1}
\]
Example: Compute the series for \( \frac{1}{(1-x)^2} \) using \( \frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \).

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\frac{d}{dx} \frac{1}{1-x} = \frac{d}{dx} (1-x)^{-1} = (-1)(1-x)^{-2}(-1)
\]
Example: Compute the series for \(\frac{1}{(1-x)^2}\) using \(\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k\).

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Example: Compute the series for $\frac{1}{(1-x)^2}$ using $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$.

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\frac{1}{(1-x)^2} = \frac{d}{dx} (1-x)^{-1} = \frac{d}{dx} \sum_{k=0}^{\infty} x^k
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\[
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\[
= \sum_{k=0}^{\infty} k x^{k-1}
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\[
= \sum_{k=0}^{\infty} kx^{k-1} \quad \text{(change summation: let} \ j = k - 1)\]
Example: Compute the series for \( \frac{1}{(1-x)^2} \) using \( \frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \).

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Thus,

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\frac{1}{(1-x)^2} = \frac{d}{dx} (1-x)^{-1} = \frac{d}{dx} \sum_{k=0}^{\infty} x^k
\]

\[
= \sum_{k=0}^{\infty} \frac{d}{dx} x^k
\]

\[
= \sum_{k=0}^{\infty} kx^{k-1} \quad \text{(change summation: let } j = k - 1) 
\]

\[
= \sum_{j=0}^{\infty} (j+1)x^j
\]
Example: Compute the series for $\frac{1}{(1-x)^2}$ using $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$.

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$$\frac{d}{dx} \frac{1}{1-x} = \frac{d}{dx} (1-x)^{-1} = (-1)(1-x)^{-2}(-1) = \frac{1}{(1-x)^2}. $$

Thus,

$$\frac{1}{(1-x)^2} = \frac{d}{dx} (1-x)^{-1} = \frac{d}{dx} \sum_{k=0}^{\infty} x^k$$

$$= \sum_{k=0}^{\infty} \frac{d}{dx} x^k$$

$$= \sum_{k=0}^{\infty} kx^{k-1} \quad \text{(change summation: let } j = k - 1 \text{)}$$

$$= \sum_{j=0}^{\infty} (j + 1)x^j = 1 + 2x + 3x^2 + 4x^3 + \cdots.$$
Example: Compute the series for \( \frac{1}{(1-x)^2} \) using \( \frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \).

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You try: Exercise 36