Math 365 – Monday 3/18/19 – 8.2: Solving linear recurrence relations

Warmup. Recall that a solution for a sequence defined recursively is a closed formula that satisfies the recursion relation and the initial conditions. For example, the sequence

$$a_n = na_{n-1},\ a_0 = 1 \qquad \text{has solution} \qquad a_n = n!.$$

(1) Consider the recurrence relation $a_n = 7a_{n-1}$.
  (a) Choose three different initial conditions (specific values for a_0), and find the corresponding solutions.
  (b) Write a general solution to this recurrence relation in terms of a_0 (without picking a specific value for a_0).

(2) For each of the following recursion relations, how many initial conditions are needed to determine a specific solution?
  (a) $a_n = 5a_{n-1}$
  (b) $a_n = 3$
  (c) $a_n = 9a_{n-2}$
  (d) $a_n = a_{n-1}^2$
  (e) $a_n = -2a_{n-1} - a_{n-2}$
  (f) $a_n = 7a_{n-1} - 6a_{n-2}$
  (g) $a_n = 8a_{n-3}$
  (h) $a_n = 3a_{n-1} + 4a_{n-2} - 12a_{n-3}$
  (i) $a_n = a_{n-1}/n$
  (j) $a_n = a_{n-1} + a_{n-2} + n + 3$

(3) Factor the following polynomials
  (a) $x^2 - 2x + 1$
  (b) $x^2 + 5x + 6$
  (c) $x^2 - 9$
  (d) $x^2 + 2x + 5$
  (e) $x^3 - 3x^2 + 3x - 1$
  (f) $x^3 - 8$
  (g) $x^3 + 5x^2 + 8x + 4$
  (h) $x^3 - 3x^2 - 4x + 12$
  (i) $x^4 - 2x^2 + 1$

Exercise 31. For each of the following, decide if the recurrence relation is linear, homogeneous, and constant coefficient. If not, explain why it fails. If so, (i) give its degree, (ii) give its characteristic equation, and (iii) give the characteristic roots and their multiplicities.

(a) $a_n = 5a_{n-1}$
(b) $a_n = 3$
(c) $a_n = 9a_{n-2}$
(d) $a_n = a_{n-1}^2$
(e) $a_n = -2a_{n-1} - a_{n-2}$
(f) $a_n = 7a_{n-1} - 6a_{n-2}$
(g) $a_n = 8a_{n-3}$
(h) $a_n = 3a_{n-1} + 4a_{n-2} - 12a_{n-3}$
(i) $a_n = a_{n-1}/n$
(j) $a_n = a_{n-1} + a_{n-2} + n + 3$

Exercise 32. For each of the recursion relations in Exercise 31 that were linear, homogeneous, and constant coefficient, decide which have characteristic equations with k distinct roots. For those that do,

(i) write a general solution;
(ii) choose an example of initial conditions, and solve for specific α_i's; and
(iii) check your answer to the previous part by computing the first 5 terms of the sequence in two ways (recursively, and using your closed formula).

Exercise 33. For each of the recursion relations in Exercise 31 that were linear, homogeneous, and constant coefficient, but had characteristic equations with repeated roots,

(i) write a general solution;
(ii) choose an example of initial conditions, and solve for specific $\alpha_{i,j}$'s; and
(iii) check your answer to the previous part by computing the first 5 terms of the sequence in two ways (recursively, and using your closed formula).

Exercise 34. Adapt the proof of Theorem 1 for $k = 2$ to prove Theorem 2 for $k = 2$. Namely, show that if

$$a_n = c_1 a_{n-1} + c_2 a_{n-2}$$

has a characteristic equation with a repeated root r_0, then $a_n = \alpha r_0^n + \beta n r_0^n$ is the general solution.

Outline/hints:

- Establish that the characteristic equation is $r^2 - c_1 r - c_2 = 0$.
- Justify that if r_0 is the only root of this equation, then actually $c_1 = 2r_0$ and $c_2 = -r_0^2$.
- Use a similar computation as in class to show that $a_n = \alpha r_0^n + \beta n r_0^n$ is a solution to the recurrence for any constants α and β.
- Use a similar computation as in class to show that if $\{a_n\}_{n \in \mathbb{N}}$ is a solution, then it must be of the form $a_n = \alpha r_0^n + \beta n r_0^n$ (i.e. there are some α and β that match your solution).

8.2: Solving linear recurrence relations

A linear homogeneous recurrence relation of degree k with constant coefficients is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k},$$

where $c_1, c_2, \ldots, c_k$ are real numbers, and $c_k \neq 0$.

Examples:

1. $a_n = a_{n-1} + a_{n-2}$ (i.e. $c_1 = c_2 = 1$.)
2. $a_n = 3a_{n-1} - a_{n-3}$ (i.e. $c_1 = 3$, $c_2 = 0$, $c_3 = -1$.)

Non-examples:

1. $a_n = a_{n-1}a_{n-2}$
2. $a_n = a_{n-1} + 1$
3. $a_n = a_{n-1}^2$
4. $a_n = na_{n-1}$

Solving recurrences in general is hard (i.e. no deterministic way to do it). We take the same approach as in solving integrals and differential equations: look at the form the recurrence takes, make an educated guess, and solve for unknowns.

Analogy to differential equations

For those who have taken 391 - otherwise ignore this slide!

In DE's, the only way we know to solve most equations is basically educated guessing. In particular, we have good guesses for *linear homogeneous constant-coefficient equations* of order k, i.e. equations of the form

$$y^{(k)} = c_1 y^{(k-1)} + c_2 y^{(k-2)} + \cdots + c_{k-2} y' + c_{k-1} y.$$

(where $y^{(i)} = \frac{d^i y}{dt^i}$)

To solve, we plug in $y = e^{rt}$ and solve for r.

Ex. Suppose

$$y'' = 5y' - 6y.$$

Plugging in $y = e^{rt}$ gives $\frac{dy}{dt} = re^{rt}$ and $\frac{d^2y}{dt^2} = r^2 e^{rt}$, so that

$$r^2 e^{rt} = 5re^{rt} - 6e^{rt}.$$

But since $e^{rt} \neq 0$, dividing through gives

$$r^2 = 5r - 6. \qquad \text{So } 0 = r^2 - 5r - 3 = (r-2)(r-3).$$

("Characteristic equation")

Thus $y_1 = e^{2t}$ and $y_2 = e^{3t}$ are both solutions.

Linearity further gives us that $y = a_1 y_1 + a_2 y_2$ is also a solution for any constants a_1 and a_2.

Back to solving recurrence relations: Educated guessing

Note that $a_n = r^n$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k},$$

if and only if (plug in $a_n = r^n$)

$$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \cdots + c_k r^{n-k}.$$

So long as $r \neq 0$ (which is always a solution), this is equivalent to

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_{k-1} r - c_k = 0 \qquad (*)$$

(subtract the RHS from both sides, and remove as many factors of r as possible).

We call this last equation the characteristic equation for the recurrence relation (same as for differential equations). The solutions to this equation are the characteristic roots.

Char. eqn.: $\quad r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_{k-1} r - c_k = 0$

For example, consider the Fibonacci sequence:

$$\boxed{a_n = a_{n-1} + a_{n-2}.}$$

Plugging in $a_n = r^n$ gives $r^n = r^{n-1} + r^{n-2}$. So

$$0 = r^n - r^{n-1} - r^{n-2} = r^{n-2}(r^2 - r - 1).$$

So the characteristic equation for this recursion relation is

$$\boxed{r^2 - r - 1 = 0.}$$

Recall the quadratic formula:

$$ax^2 + bx + c = 0 \quad \text{if and only if } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

This gives that the characteristic roots are

$$r_1 = \tfrac{1}{2}\left(1 + \sqrt{5}\right) \quad \text{and} \quad r_2 = \tfrac{1}{2}\left(1 - \sqrt{5}\right)$$

(each with multiplicity 1).

Char. eqn.: $r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_{k-1} r - c_k = 0$

As another example, consider the recursion relation

$$a_n = -a_{n-1} + a_{n-2} + a_{n-3}.$$

You try: plug in $a_n = r^n$ and simplify.

Characteristic equation:

Theorem 1: Solving linear homogeneous with distinct roots

Let $c_1, c_2, \dots, c_k$ be real numbers. Suppose that the characteristic equation has k distinct roots $r_1, r_2, \dots, r_k$. Then a sequence $\{a_n\}_{n\in\mathbb{N}}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$ if and only if

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \cdots + \alpha_k r_k^n$$

for $n = 0, 1, 2, \dots$, where $\alpha_1, \alpha_2, \dots, \alpha_k$ are constants.

Example: For $a_n = a_{n-1} + a_{n-2}$, we found characteristic roots

$$r_1 = \tfrac{1}{2}\left(1+\sqrt{5}\right) \quad \text{and} \quad r_2 = \tfrac{1}{2}\left(1-\sqrt{5}\right),$$

each with multiplicity 1. So the general solution is

$$a_n = \alpha_1 \left(\tfrac{1}{2}\left(1+\sqrt{5}\right)\right)^n + \alpha_2 \left(\tfrac{1}{2}\left(1-\sqrt{5}\right)\right)^n.$$

Example: For $a_n = -a_{n-1} + a_{n-2} + a_{n-3}$, $r_2 = -1$ had multiplicity 2, so this theorem does not apply!

Incorporating initial conditions: specific solutions

Now, we have a general solution to the recurrence relation $a_n = a_{n-1} + a_{n-2}$:

$$a_n = \alpha_1 \left(\tfrac{1}{2}\left(1+\sqrt{5}\right)\right)^n + \alpha_2 \left(\tfrac{1}{2}\left(1-\sqrt{5}\right)\right)^n.$$

Initial conditions: For example, suppose we have $a_0 = 0$ and $a_1 = 1$. To solve for the *specific solution*, just plug those values into the general solution and solve for the unknowns.

$$0 = a_0 = \alpha_1 \left(\tfrac{1}{2}\left(1+\sqrt{5}\right)\right)^0 + \alpha_2 \left(\tfrac{1}{2}\left(1-\sqrt{5}\right)\right)^0 = \alpha_1 + \alpha_2;$$

$$1 = a_1 = \alpha_1 \left(\tfrac{1}{2}\left(1+\sqrt{5}\right)\right)^1 + \alpha_2 \left(\tfrac{1}{2}\left(1-\sqrt{5}\right)\right)^1$$

$$= \alpha_1 \left(\tfrac{1}{2}\left(1+\sqrt{5}\right)\right) - \alpha_1 \left(\tfrac{1}{2}\left(1-\sqrt{5}\right)\right) = \alpha_1\sqrt{5}.$$

So $\alpha_1 = 1/\sqrt{5}$ and $\alpha_2 = -1/\sqrt{5}$. Therefore

$$a_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

You try Exercise 32

Theorem 2: Solving linear homogeneous with repeated roots

Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose that the characteristic equation has roots $r_1, r_2, \ldots, r_\ell$ with multiplicities $m_1, m_2, \ldots, m_\ell$. Then a sequence $\{a_n\}_{n\in\mathbb{N}}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$ if and only if

$$a_n = p_1(n) r_1^n + p_2(n) r_2^n + \cdots + p_\ell(n) r_\ell^n,$$

where $p_i(n)$ are polynomials in n of degree $m_i - 1$.

Example: $a_n = -a_{n-1} + a_{n-2} + a_{n-3}$.

Characteristic equation:

$$0 = r^3 + r^2 - r - 1 = (r+1)^2 (r-1)^1.$$

General solution:

$$a_n = (\alpha_0 + \alpha_1 n)(-1)^n + \beta(1)^n.$$

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where $p_i(n)$ are polynomials in n of degree $m_i - 1$.

Think of Theorems 1 and 2 as recipes for cooking up solutions!

Goal: You need k servings of solutions.

1. If you have enough ingredients from the char. eq., then just use those. (Thm 1)
2. If the char. eq. didn't give you enough, you'll need to make some more first: (Thm 2)
   - if a root r was repeated m times, you'll need to stretch it for m servings by multiplying it by n successively until you have enough.

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Example: $a_n = 5a_{n-1} + 5a_{n-2} - 25a_{n-3} - 40a_{n-4} - 16a_{n-5}$.
Characteristic equation:

$$0 = r^5 - 5r^4 - 5r^3 + 25r^2 + 40r + 16 = (r-4)^2(r+1)^3.$$

Here, $r_1 = 4$ needs to cover two solutions, so we'll stretch it by using solutions

$$4^n \qquad \text{and} \qquad n4^n;$$

and $r_2 = -1$ needs to cover three solutions, so we'll stretch it by using solutions

$$(-1)^n, \qquad n(-1)^n, \quad \text{and} \quad n^2(-1)^n.$$

General solution: (simplified)

$$a_n = (\alpha_0 + \alpha_1 n)(4)^n + (\beta_0 + \beta_1 n + \beta_2 n^2)(-1)^n.$$

You try Exercise 33

Let's see why:

Proof of Theorem 1 for $k = 2$. Consider the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2},$$

and suppose the corresponding characteristic equation $r^2 - c_1 r - c_2 = 0$ has distinct roots r_1 and r_2. Note that this means that

$$r_1^2 = c_1 r_1 + c_2 \quad \text{and} \quad r_2^2 = c_1 r_2 + c_2.$$

First let's see that $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ is a solution for any α_1, α_2:

$$\begin{aligned} c_1 a_{n-1} + c_2 a_{n-2} &= c_1(\alpha_1 r_1^{n-1} + \alpha_2 r_2^{n-1}) + c_2(\alpha_1 r_1^{n-2} + \alpha_2 r_2^{n-2}) \\ &= \alpha_1 r_1^{n-2}(c_1 r_1 + c_2) + \alpha_2 r_2^{n-2}(c_1 r_2 + c_2) \\ &= \alpha_1 r_1^{n-2}(r_1^2) + \alpha_2 r_2^{n-2}(r_2^2) \\ &= \alpha_1 r_1^n + \alpha_2 r_2^n \\ &= a_n. \qquad \checkmark \end{aligned}$$

Proof of Theorem 1 for $k = 2$ continued:

Now, we have to show that *every* solution to $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ is of the form $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$. To do this, note that

the recurrence relation, together with "enough" initial conditions *totally determines the sequence!*

So, take any solution $\{a_n\}_{n \in \mathbb{N}}$ to this recursion relation. Whatever a_0 and a_1 are, those are the initial conditions.

To do*: show that there is some α_1 and α_2 such that

$$a_0 = \alpha_1 r_1^0 + \alpha_2 r_2^0 \text{ and } a_1 = \alpha_1 r_1^1 + \alpha_2 r_2^1.$$

Conclusion: Thus, since $\alpha_1 r_1^n + \alpha_2 r_2^n$ is a solution (by what we just did) and satisfies the same initial conditions, it must be the same solution as the one we picked.

*: Solve

$$a_0 = \alpha_1 + \alpha_2 \qquad \text{and} \qquad a_1 = \alpha_1 r_1 + \alpha_2 r_2$$

for α_1 and α_2.

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$$a_0 = \alpha_1 + \alpha_2 \qquad \text{and} \qquad a_1 = \alpha_1 r_1 + \alpha_2 r_2$$

for α_1 and α_2:

The first equation gives $\alpha_2 = a_0 - \alpha_1$.

Substitute this into the second equation to get

$$a_1 = \alpha_1 r_1 + (a_0 - \alpha_1) r_2 = \alpha_1 (r_1 - r_2) + a_0 r_2.$$

So as long as $r_1 \neq r_2$, we have

$$\alpha_1 = \frac{a_1 - a_0 r_2}{r_1 - r_2} \qquad \text{and} \qquad \alpha_2 = a_0 - \frac{a_1 - a_0 r_2}{r_1 - r_2}.$$

This completes our proof of Theorem 1 for $k = 2$.