Common errors on the quizzes

1. Define “cardinality”: *Two sets* $X$ and $Y$ *have the same cardinality* if...
   
   [Hint: “…they have the same size” is not the right answer.]

   $X$ and $Y$ have the same number of elements
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Problem: what about $\mathbb{R}$ versus $\mathbb{Z}$? Or $\mathbb{Z}$ versus $\mathbb{Z}_{\geq 1}$?
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\[ X \text{ and } Y \text{ have the same number of elements} \]

**Problem:** what about \( \mathbb{R} \) versus \( \mathbb{Z} \)? Or \( \mathbb{Z} \) versus \( \mathbb{Z}_{\geq 1} \)?

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**Good:**

1. Define “cardinality”: *Two sets* \( X \) and \( Y \) *have the same cardinality if*...
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\[ X \text{ and } Y \text{ are in bijection with eachother.} \]

1. Define “cardinality”: *Two sets* \( X \) and \( Y \) *have the same cardinality if*...
   [Hint: “...they have the same size” is not the right answer.]

\[ \text{there is a bijective function } f : X \rightarrow Y. \]
"Outline a proof by induction that $\sum_{i=1}^{n} i = n(n + 1)/2$.

**Inductive step**

\[
\sum_{i=1}^{n+1} i = \frac{(n+1)(n+2)}{2} - (n+1)
\]

\[
1 + 2 + \ldots + n + n+1 = \frac{(n+1)(n+2)}{2} - (n+1)
\]

\[
x^2: 2 \left( 1 + 2 + \ldots + n \right) = (n+1)(n+2) - 2(n+1)
\]

\[
2 \sum_{i=1}^{n} i = n^2 + 3n + 2 - n - 2
\]

\[
2 \left( \frac{(n+1)n}{2} \right) = n^2 + 2n
\]

\[
(n+1)n = n^2 + 2n \quad \Box
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Outline a proof by induction that $\sum_{i=1}^{n} = n(n + 1)/2$.

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\[
(n+1)n = n^2 + 2n \quad \checkmark
\]

**Problem:** Manipulating formulas from “what you want” to “something true” isn’t very reliable.
Bad example.

Claim: \(-1 = 1\)
Bad example.

Claim: \(-1 = 1\)

Non-proof.
If \(-1 = 1\), then
\[
(-1)^2 = (1)^2
\]
Bad example.

**Claim:** $-1 = 1$

**Non-proof.**

If $-1 = 1$, then

$(-1)^2 = (1)^2$, so that $1 = 1$,

which is true.
Bad example.

Claim: $-1 = 1$

Non-proof.

\[
\begin{align*}
\text{If } -1 &= 1, \\
(-1)^2 &= (1)^2, \quad \text{so that } \quad 1 &= 1, \\
\end{align*}
\]

which is true.

What went wrong:

We proved that

\[\text{“} -1 = 1 \text{ implies } 1 = 1 \text{”}, \]

which is (strangely enough) true. A false statement can imply a true statement.

We did not show that $-1 = 1$. 

“Outline a proof by induction that \( \sum_{i=1}^{n} i = n(n + 1)/2 \).

**Inductive step**

\[
\sum_{i=1}^{n+1} i = \frac{(n+1)(n+2)}{2} - (n+1)
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\[\times 2: \quad 2\left(1+2+ \ldots +n\right) = (n+1)(n+2) - 2(n+1)\]

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\[(n+1)n = n^2 + 2n \quad \checkmark\]

**Problem:** Manipulating formulas from “what you want” to “something true” isn’t very reliable.
“Outline a proof by induction that $\sum_{i=1}^{n} = n(n + 1)/2$.

Good:

- **Inductive Step:** Fix $n \geq 1$, and assume
  \[
  \sum_{i=1}^{n} = \frac{n(n+1)}{2}
  \]
  (for that $n$).

  Then
  \[
  \sum_{i=1}^{n+1} i = 1 + 2 + \cdots + n + (n+1)
  \]
  
  \[
  = \frac{n(n+1)}{2} + (n+1) \quad \text{(by the induction hypothesis)}
  \]
  
  \[
  = \frac{n^2 + n + 2n + 2}{2} = \frac{(n+1)(n+2)}{2}.
  \]
“Outline a proof by induction that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

One more problem:

$P(n) = \frac{\sum_{i=1}^{n} i}{2} = \frac{n(n+1)}{2}$ versus

$P(n) : \frac{\sum_{i=1}^{n} i}{2} = \frac{n(n+1)}{2}$
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One more problem:

$\mathcal{P}(n) = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ versus $\mathcal{P}(n): \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

Problem: $\mathcal{P}(n)$ is a statement, like “avocados are green”, and can’t equal anything.

On the other hand, $n(n + 1)/2$ is an expression, which is basically a noun, like “avocados”.
Last time: Counting rules.

**Product:** If a procedure can be broken into a sequence of two tasks, and there are $n_1$ ways to do the first task, and for each of these ways of doing the first task, there are $n_2$ ways to doing the second task, then there are $n_1n_2$ total distinct outcomes.

**Sum:** If a procedure can be done either in one of $n_1$ ways or in one of $n_2$ ways, where there is no overlap in the $n_1$ and $n_2$ ways, then there are $n_1 + n_2$ total distinct outcomes.

**Subtraction/Inclusion-exclusion:** If a procedure can be done either in one of $n_1$ ways or in one of $n_2$ ways, but there are $n_3$ overlapping outcomes, then there are $n_1 + n_2 - n_3$ total distinct outcomes.

**Division:** If a procedure can be done in $n$ ways, but that procedure produces each outcome in $d$ different ways, then there are actually $n/d$ distinct outcomes.
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**Remember:** Every rule depends on making up a “procedure” for counting, and then applying the rules according to that procedure!! (Take it from this expert: *Never* just plug stuff into a formula – make up a story for counting things one step at a time, and *then* try to count.)
The choose function is

\[ C(n, k) = \binom{n}{k} = \#\{ \text{ways to choose } k \text{ objects from } n \} \],

read “n choose k”.

Now that we have some counting skills, we can build a formula using the product and division rules. Let’s start with an example.
The choose function is

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Let's start with an example: $n = 7$, $k = 3$.

**Step 1:** How many ways are there to select 3 things from 7 in order (no replacement)?
\[ C(n, k) = \binom{n}{k} = \#\{ \text{ways to choose } k \text{ objects from } n \} \]

Let's start with an example: \( n = 7, \ k = 3 \).

**Step 1:** How many ways are there to select 3 things from 7 in order (no replacement)?

**Ans:** \( 7 \ast 6 \ast 5 \).
Let's start with an example: $n = 7$, $k = 3$.

**Step 1:** How many ways are there to select 3 things from 7 in order (no replacement)?

**Ans:** $7 \times 6 \times 5$.

Rewriting, notice that

$$7! = (7 \times 6 \times 5) \times \frac{(4 \times 3 \times 2 \times 1)}{(7-3)!}$$

where $(7-3)! = 4!$. 

$$C(n, k) = \binom{n}{k} = \# \{ \text{ways to choose } k \text{ objects from } n \}$$
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\[ 7 \times 6 \times 5 = \frac{7!}{(7 - 3)!} . \]
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**Step 2:** How many ways are there to put the 3 things in order? (How many permutations are there of 3 things?)
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**Ans:** \( 3 \times 2 \times 1 = 3! \).
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So \[ C(7, 3) = \frac{7!}{(7-3)!3!}. \]
\[ C(n, k) = \binom{n}{k} = \#\{ \text{ways to choose } k \text{ objects from } n \} \]

Now for general \( n \geq k \geq 1 \):

**Step 1:** How many ways are there to select \( k \) things from \( n \) in order (no replacement)?
\[ C(n, k) = \binom{n}{k} = \# \{ \text{ways to choose } k \text{ objects from } n \} \]

Now for general \( n \geq k \geq 1 \):

Step 1: How many ways are there to select \( k \) things from \( n \) in order (no replacement)?

Ans: \( n * (n - 1) * (n - 2) \cdots (n - (k - 1)) \).
\[ C(n, k) = \binom{n}{k} = \#\{ \text{ways to choose } k \text{ objects from } n \} \]

Now for general \( n \geq k \geq 1 \):

**Step 1:** How many ways are there to select \( k \) things from \( n \) in order (no replacement)?

**Ans:** \( n \ast (n - 1) \ast (n - 2) \cdots (n - (k - 1)) \).

Rewriting, notice that

\[
n! = (n\ast(n-1)\cdots(n-(k-1)))\ast((n-k)\ast(n-k-1)\cdots2\ast1) \left/ (n-k)! \right.
\]
\[ C(n, k) = \binom{n}{k} = \#\{ \text{ways to choose } k \text{ objects from } n \} \]

Now for general \( n \geq k \geq 1 \):

**Step 1:** How many ways are there to select \( k \) things from \( n \) in order (no replacement)?

**Ans:** \( n \times (n - 1) \times (n - 2) \cdots (n - (k - 1)) \).

Rewriting, notice that

\[
n! = \left( n \times (n - 1) \cdots (n - (k - 1)) \right) \times \left( (n - k) \times (n - k - 1) \cdots \times 2 \times 1 \right),
\]

so

\[
n \times (n - 1) \times (n - 2) \cdots (n - (k - 1)) = \frac{n!}{(n - k)!}.
\]
\[ \binom{n}{k} = \# \{ \text{ways to choose } k \text{ objects from } n \} \]

Now for general \( n \geq k \geq 1 \):

**Step 1:** There are \( n!/(n - k)! \) ways to select \( k \) things from \( n \) in order.
\[ \binom{n}{k} = \#\{ \text{ways to choose } k \text{ objects from } n \} \]

Now for general \( n \geq k \geq 1 \):

**Step 1:** There are \( n!/(n - k)! \) ways to select \( k \) things from \( n \) in order.

**Step 2:** How many ways are there to put the \( k \) things in order?
(How many permutations are there of \( k \) things?)
\begin{align*}
\binom{n}{k} &= \#\{ \text{ ways to choose } k \text{ objects from } n \} \\
\text{Now for general } n \geq k \geq 1: \\
\text{Step 1: There are } n!/(n - k)! \text{ ways to select } k \text{ things from } n \text{ in order.} \\
\text{Step 2: How many ways are there to put the } k \text{ things in order?} \\
\text{(How many permutations are there of } k \text{ things?)} \\
\text{Ans: } k \cdot (k - 1) \cdot (k - 2) \cdot \cdots \cdot 2 \cdot 1 = k!.
\end{align*}
\[
{n \choose k} = \#\{ \text{ ways to choose } k \text{ objects from } n \} 
\]

Now for general \( n \geq k \geq 1 \):

**Step 1:** There are \( n!/(n-k)! \) ways to select \( k \) things from \( n \) in order.

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\textbf{Ans: } \( k \times (k-1) \times (k-2) \times \cdots \times 2 \times 1 = k! \).

**Step 3:** Use the division rule to combine step 1 and step 2.
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\binom{n}{k} = \#\{ \text{ways to choose } k \text{ objects from } n \} 
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Now for general \( n \geq k \geq 1 \):

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\text{Ans: } k \cdot (k-1) \cdot (k-2) \cdot \cdots \cdot 2 \cdot 1 = k!.
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**Step 3:** Use the division rule to combine step 1 and step 2. Namely, the number of ways to choose \( k \) things \textbf{without order} from \( n \) is

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\frac{\text{step 1}}{\text{step 2}}
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\binom{n}{k} = \# \{ \text{ ways to choose } k \text{ objects from } n \} 
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Now for general \( n \geq k \geq 1 \):

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\frac{\text{step 1}}{\text{step 2}} = \frac{n!/(n-k)!}{k!}
\]
$$\binom{n}{k} = \#\{\text{ways to choose } k \text{ objects from } n \}$$

Now for general $n \geq k \geq 1$:

**Step 1:** There are $n!/(n - k)!$ ways to select $k$ things from $n$ in order.

**Step 2:** How many ways are there to put the $k$ things in order? (How many permutations are there of $k$ things?)

**Ans:** $k \times (k - 1) \times (k - 2) \times \cdots \times 2 \times 1 = k!$.

**Step 3:** Use the division rule to combine step 1 and step 2. Namely, the number of ways to choose $k$ things without order from $n$ is

$$\left(\frac{\text{step 1}}{\text{step 2}}\right) = \frac{n!/(n - k)!}{k!} = \frac{n!}{(n - k)!k!}.$$
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Now for general \( n \geq k \geq 1:\)

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\]

So \[
C(n, k) = \frac{n!}{(n-k)!k!}.
\]
\[ C(n, k) = \binom{n}{k} = \#\{ \text{ ways to choose } k \text{ objects from } n \} = \frac{n!}{(n-k)!k!} \]

A \textit{k-permutation} of \( n \) objects is a choice of \( k \) things from \( n \) \textbf{in order}.
\[ C(n, k) = \binom{n}{k} = \#\{ \text{ ways to choose } k \text{ objects from } n \} = \frac{n!}{(n-k)!k!} \]

A \textit{k-permutation} of \( n \) objects is a choice of \( k \) things from \( n \) \textbf{in order}. The \textit{permutation} function is

\[ P(n, k) = \#\{ \text{ ways to select } k \text{ objects from } n \text{ in order} \}. \]
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As we saw,

\[ P(n, k) = \frac{n!}{(n - k)!}. \]
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A \textit{permutation} is an \( n \)-permutation of \( n \) objects.

We call counting problems that call for \underline{unordered} selection “\underline{combination problems}”.

We call counting problems that call for \underline{ordered} selection “\underline{permutation problems}”. 
\[ C(n, k) = \binom{n}{k} = \#\{ \text{ways to choose } k \text{ objects from } n \} = \frac{n!}{(n-k)!k!} \]

\[ P(n, k) = \#\{ \text{ways to select } k \text{ objects from } n \text{ in order} \} = \frac{n!}{(n-k)!} \]

Some examples:
\[ C(n, k) = \binom{n}{k} = \#\{ \text{ways to choose} \ k \ \text{objects from} \ n \} = \frac{n!}{(n-k)!k!} \]

\[ P(n, k) = \#\{ \text{ways to select} \ k \ \text{objects from} \ n \ \text{in order} \} = \frac{n!}{(n-k)!} \]

Some examples:

(1) How many ways can one pick a president, vice president, and secretary in a club of 20 people?
\[ C(n, k) = \binom{n}{k} = \#\{ \text{ways to choose } k \text{ objects from } n \} = \frac{n!}{(n-k)!k!} \]
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Some examples:

(1) How many ways can one pick a president, vice president, and secretary in a club of 20 people? (Permutation, \( n = 20, k = 3 \))
\[ C(n, k) = \binom{n}{k} = \#\{ \text{ways to choose } k \text{ objects from } n \} = \frac{n!}{(n-k)!k!} \]
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Some examples:

(1) How many ways can one pick a president, vice president, and secretary in a club of 20 people? (Permutation, \( n = 20, k = 3 \))
\[
P(20, 3) = 20 \times 19 \times 18.
\]
\[ C(n, k) = \binom{n}{k} = \#\{ \text{ways to choose} \; k \; \text{objects from} \; n \} = \frac{n!}{(n-k)!k!} \]

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Some examples:

(1) How many ways can one pick a president, vice president, and secretary in a club of 20 people? (Permutation, \( n = 20, \; k = 3 \))

\[ P(20, 3) = 20 \times 19 \times 18. \]

(2) How many ways can one pick a committee of 3 from a club of 20 people?
\[
C(n, k) = \binom{n}{k} = \#\{ \text{ways to choose } k \text{ objects from } n \} = \frac{n!}{(n-k)!k!}
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\]

Some examples:

(1) How many ways can one pick a president, vice president, and secretary in a club of 20 people? (**Permutation**, \(n = 20, k = 3\))
\[
\boxed{P(20, 3) = 20 \times 19 \times 18.}
\]

(2) How many ways can one pick a committee of 3 from a club of 20 people? (**Combination**, \(n = 20, k = 3\))
\[ C(n, k) = \binom{n}{k} = \# \{ \text{ways to choose } k \text{ objects from } n \} = \frac{n!}{(n-k)!k!} \]
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Some examples:

(1) How many ways can one pick a president, vice president, and secretary in a club of 20 people? (Permutation, \( n = 20, \ k = 3 \))
\[
P(20, 3) = 20 \times 19 \times 18.
\]

(2) How many ways can one pick a committee of 3 from a club of 20 people? (Combination, \( n = 20, \ k = 3 \))
\[
C(20, 3) = 20 \times 19 \times 18/(3 \times 2 \times 1).
\]
\[ C(n, k) = \binom{n}{k} = \#\{ \text{ways to choose } k \text{ objects from } n \} = \frac{n!}{(n-k)!k!} \]
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Some examples:

(1) How many ways can one pick a president, vice president, and secretary in a club of 20 people? (Permutation, \( n = 20, \ k = 3 \))

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\[
C(20, 3) = \frac{20 \times 19 \times 18}{(3 \times 2 \times 1)}.\
\]

(3) A coin is flipped 5 times. How many ways could it turn out that heads comes up exactly 3 times?
\begin{align*}
C(n, k) &= \binom{n}{k} = \# \{ \text{ways to choose } k \text{ objects from } n \} = \frac{n!}{(n-k)!k!} \\
P(n, k) &= \# \{ \text{ways to select } k \text{ objects from } n \text{ in order} \} = \frac{n!}{(n-k)!}
\end{align*}

Some examples:

(1) How many ways can one pick a president, vice president, and secretary in a club of 20 people? (Permutation, \(n = 20, k = 3\))

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P(20, 3) = 20 \times 19 \times 18.
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(2) How many ways can one pick a committee of 3 from a club of 20 people? (Combination, \(n = 20, k = 3\))

\[
C(20, 3) = \frac{20 \times 19 \times 18}{3 \times 2 \times 1}.
\]

(3) A coin is flipped 5 times. How many ways could it turn out that heads comes up exactly 3 times?

Ans: When I choose the 3 times that the heads come up, it doesn’t matter what order I choose them, just which slots I pick.
\[ C(n, k) = \binom{n}{k} = \#\{ \text{ways to choose } k \text{ objects from } n \} = \frac{n!}{(n-k)!k!} \]
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Some examples:

(1) How many ways can one pick a president, vice president, and secretary in a club of 20 people? (Permutation, \( n = 20 \), \( k = 3 \))
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Some examples:

(1) How many ways can one pick a president, vice president, and secretary in a club of 20 people? (Permutation, \( n = 20, \ k = 3 \))

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\[
C(20, 3) = 20 \times 19 \times 18/(3 \times 2 \times 1).
\]

(3) A coin is flipped 5 times. How many ways could it turn out that heads comes up exactly 3 times?

Ans: When I choose the 3 times that the heads come up, it doesn’t matter what order I choose them, just which slots I pick. (Combination, \( n = 5, \ k = 3 \))

\[
C(5, 3) = 5 \times 4 \times 3/(3 \times 2 \times 1).
\]
\[ C(n, k) = \binom{n}{k} = \#\{ \text{ways to choose } k \text{ objects from } n \} = \frac{n!}{(n-k)!k!} \]
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Some examples:

(4) True or false problem from the in-class exercises last time:
   How many ways can a student answer a True or False quiz with 10 questions if they may or may not leave a problem blank?
\[ C(n, k) = \binom{n}{k} = \#\{ \text{ways to choose } k \text{ objects from } n \} = \frac{n!}{(n-k)!k!} \]

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(4) True or false problem from the in-class exercises last time:
How many ways can a student answer a True or False quiz with 10 questions if they may or may not leave a problem blank?

(Counting in two different ways gives the identity

\[ 3^{10} = \sum_{i=0}^{10} \binom{10}{i} 2^{10-i}. \]
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\[ 3^{10} = \sum_{i=0}^{10} \binom{10}{i} 2^{10-i} \])

You try: Exercise 22
The Pigeonhole Principle says that if \( k + 1 \) objects are placed into \( k \) boxes, then at least one box contains two or more objects.
Pigeonhole principle - §6.2

The Pigeonhole Principle says that if \( k + 1 \) objects are placed into \( k \) boxes, then at least one box contains two or more objects.

Proof (by contradiction). Suppose every box contains at most one object. Then there are at most \( k \) boxes with at most \( 1 \) object per box. This is a contradiction. So at least one box contains two or more objects.

In the language of functions: If \( A \) is a set of size \( k + 1 \) and \( B \) is a set of size \( k \), then there is no injective function \( f: A \rightarrow B \). (It follows that if \( |A| \geq |B| \), then there is no injective function \( f: A \rightarrow B \).)
The Pigeonhole Principle says that if \( k + 1 \) objects are placed into \( k \) boxes, then at least one box contains two or more objects.

Proof (by contradiction).
Suppose every box contains at most one object.
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Suppose every box contains at most one object. Then there are at most

\[
\#\{ \text{boxes} \} \times (\text{max } \# \text{ objects per box})
\]
The Pigeonhole Principle says that if \( k + 1 \) objects are placed into \( k \) boxes, then at least one box contains two or more objects.

Proof (by contradiction).
Suppose every box contains at most one object. Then there are at most

\[
\#\{ \text{boxes} \} \times (\text{max # objects per box}) = k \times 1
\]
The Pigeonhole Principle says that if $k + 1$ objects are placed into $k$ boxes, then at least one box contains two or more objects.

Proof (by contradiction).

Suppose every box contains at most one object. Then there are at most

$$\#\{ \text{boxes} \} \times (\max \# \text{ objects per box } ) = k \times 1 < k + 1.$$
The Pigeonhole Principle says that if $k + 1$ objects are placed into $k$ boxes, then at least one box contains two or more objects.

**Proof (by contradiction).**

Suppose every box contains at most one object. Then there are at most

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This is a contradiction. So at least one box contains two or more objects.
Pigeonhole principle - §6.2

The Pigeonhole Principle says that if \( k + 1 \) objects are placed into \( k \) boxes, then at least one box contains two or more objects.

\[ ??? \]

\[
\begin{array}{ccc}
\text{Box 1} & \text{Box 2} & \text{Box 3} \\
\text{Object} & \text{Object} & \text{Object} \\
\end{array}
\]

Proof (by contradiction).

Suppose every box contains at most one object. Then there are at most

\[ \#\{ \text{boxes} \} \times (\text{max \# objects per box}) = k \times 1 < k + 1. \]

This is a contradiction. So least one box contains two or more objects.

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If \( A \) is a set of size \( k + 1 \) and \( B \) is a set of size \( k \), then there is no injective function \( f : A \to B \). (It follows that if \( |A| > |B| \), then there is no injective function \( f : A \to B \).)
**Pigeonhole Principle:** “if \( k + 1 \) objects are placed into \( k \) boxes, then at least one box contains two or more objects.”

**Ex:** In any group of 367 people, at least two of those people have the same birthday.
Pigeonhole Principle: “if \( k + 1 \) objects are placed into \( k \) boxes, then at least one box contains two or more objects.”

**Ex:** In any group of 367 people, at least two of those people have the same birthday.

**NON-Ex:** In a group of 367 people, it is not guaranteed that at least two people were born on a *specific* day.
Pigeonhole Principle: “if \( k + 1 \) objects are placed into \( k \) boxes, then at least one box contains two or more objects.”

**Ex:** In any group of 367 people, at least two of those people have the same birthday.

**NON-Ex:** In a group of 367 people, it is not guaranteed that at least two people were born on a specific day. For example, it is not guaranteed that at least two people were born on January 1st. (It is not guaranteed that any of them were born on January 1st!)
**Pigeonhole Principle:** “if \( k + 1 \) objects are placed into \( k \) boxes, then at least one box contains two or more objects.”

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**Ex:** In any set of 27 english words, at least two start with the same letter; at least two end with the same letter.
Pigeonhole Principle: “if \( k + 1 \) objects are placed into \( k \) boxes, then at least one box contains two or more objects.”

**Ex:** In any group of 367 people, at least two of those people have the same birthday.

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**Ex:** In any set of 27 English words, at least two start with the same letter; at least two end with the same letter.

**NON-Ex:** In a set of twenty seven English words, it is not guaranteed that at least two start with a *specific* letter.
Pigeonhole Principle: “if $k + 1$ objects are placed into $k$ boxes, then at least one box contains two or more objects.”

**Ex:** In any group of 367 people, at least two of those people have the same birthday.

**NON-Ex:** In a group of 367 people, it is not guaranteed that at least two people were born on a *specific* day. For example, it is not guaranteed that at least two people were born on January 1st. (It is not guaranteed that *any* of them were born on January 1st!)

**Ex:** In any set of 27 English words, at least two start with the same letter; at least two end with the same letter.

**NON-Ex:** In a set of twenty seven English words, it is not guaranteed that at least two start with a *specific* letter. For example, it is not guaranteed that at least two start with ‘z’ (say, the twenty seven distinct words in this example).
The Generalized Pigeonhole Principle says that if \( N \) objects are placed into \( k \) boxes, then at least one box contains \( \lceil N/k \rceil \) objects.
The Generalized Pigeonhole Principle says that if \( N \) objects are placed into \( k \) boxes, then at least one box contains \( \lceil N/k \rceil \) objects.

Proof (by contradiction).

Note that

\[
\lceil N/k \rceil < (N/k) + 1
\]

(the ceiling function rounds up, which increases a value by less than 1).
The Generalized Pigeonhole Principle says that if \( N \) objects are placed into \( k \) boxes, then at least one box contains \( \lceil N/k \rceil \) objects.

Proof (by contradiction).

Note that

\[
\lceil N/k \rceil < \frac{N}{k} + 1
\]

(the ceiling function rounds up, which increases a value by less than 1). So, multiplying both sides by \( k \), we get

\[
k\lceil N/k \rceil < k\left(\frac{N}{k} + 1\right)
\]
The Generalized Pigeonhole Principle says that if $N$ objects are placed into $k$ boxes, then at least one box contains $\lceil N/k \rceil$ objects.

Proof (by contradiction).

Note that

$$\lceil N/k \rceil < (N/k) + 1$$

(the ceiling function rounds up, which increases a value by less than 1). So, multiplying both sides by $k$, we get

$$k\lceil N/k \rceil < k((N/k) + 1) = N + k.$$
The **Generalized Pigeonhole Principle** says that if $N$ objects are placed into $k$ boxes, then at least one box contains $\lceil N/k \rceil$ objects.

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$$k\lceil N/k \rceil < k((N/k) + 1) = N + k.$$  

Now suppose every box contains at most $\lceil N/k \rceil - 1$ objects (note $\lceil N/k \rceil \geq 1$).
The Generalized Pigeonhole Principle says that if \( N \) objects are placed into \( k \) boxes, then at least one box contains \( \lceil N/k \rceil \) objects.

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\[
\#\{ \text{boxes} \} \ast (\max \# \text{ objects per box })
\]
The Generalized Pigeonhole Principle says that if \( N \) objects are placed into \( k \) boxes, then at least one box contains \( \lceil N/k \rceil \) objects.

Proof (by contradiction).

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$$\#\{\text{boxes}\} \times (\max \# \text{ objects per box } ) = k \times (\lceil N/k \rceil - 1)$$

$$= k\lceil N/k \rceil - k.$$
The Generalized Pigeonhole Principle says that if \( N \) objects are placed into \( k \) boxes, then at least one box contains \( \lceil N/k \rceil \) objects.

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The Generalized Pigeonhole Principle says that if $N$ objects are placed into $k$ boxes, then at least one box contains $\lceil N/k \rceil$ objects.

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Note that

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Now suppose every box contains at most $\lceil N/k \rceil - 1$ objects (note $\lceil N/k \rceil \geq 1$). Then there are at most

$$\#\{\text{boxes}\} \times (\text{max \ # objects per box}) = k \times (\lceil N/k \rceil - 1)$$

$$= k\lceil N/k \rceil - k < (N + k) - k = N.$$ 

This is a contradiction. So least one box contains $\lceil N/k \rceil$ or more objects. \qed
The **Generalized Pigeonhole Principle** says that if $N$ objects are placed into $k$ boxes, then at least one box contains $\lceil N/k \rceil$ objects.

**Example:** What is the minimum number of students required in a class to ensure that at least ten people will receive the same grade (if the grade options are just A,B,C,D,F)?
The **Generalized Pigeonhole Principle** says that if $N$ objects are placed into $k$ boxes, then at least one box contains $\lceil N/k \rceil$ objects.

**Example:** What is the minimum number of students required in a class to ensure that at least ten people will receive the same grade (if the grade options are just A,B,C,D,F)?

**Answer:**
Here, the grades are the “boxes”, of which there are 5. ($k = 5$).
The **Generalized Pigeonhole Principle** says that if $N$ objects are placed into $k$ boxes, then at least one box contains $\left\lfloor \frac{N}{k} \right\rfloor$ objects.

---

**Example:** What is the minimum number of students required in a class to ensure that at least ten people will receive the same grade (if the grade options are just A, B, C, D, F)?

**Answer:**
Here, the grades are the “boxes”, of which there are 5. ($k = 5$). Need at least one of the boxes to have at least ten students.
The Generalized Pigeonhole Principle says that if $N$ objects are placed into $k$ boxes, then at least one box contains $\lfloor N/k \rfloor$ objects.

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Here, the grades are the “boxes”, of which there are 5. ($k = 5$). Need at least one of the boxes to have at least ten students. ($\lfloor N/5 \rfloor \geq 10$)
The Generalized Pigeonhole Principle says that if \( N \) objects are placed into \( k \) boxes, then at least one box contains \( \lceil N/k \rceil \) objects.

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**Answer:**
Here, the grades are the “boxes”, of which there are 5. (\( k = 5 \)). Need at least one of the boxes to have at least ten students. (\( \lceil N/5 \rceil \geq 10 \))

But

\[
\lceil N/5 \rceil \geq 10 \quad \text{exactly when} \quad N/5 > 9.
\]
The Generalized Pigeonhole Principle says that if \( N \) objects are placed into \( k \) boxes, then at least one box contains \( \lfloor N/k \rfloor \) objects.

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Here, the grades are the “boxes”, of which there are 5. \( (k = 5) \).
Need at least one of the boxes to have at least ten students. \( ([N/5] \geq 10) \)
But
\[
[N/5] \geq 10 \quad \text{exactly when} \quad N/5 > 9.
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So the question comes down to finding the least integer \( N \) such that \( N > 5 \times 9 \).
The **Generalized Pigeonhole Principle** says that if $N$ objects are placed into $k$ boxes, then at least one box contains $\lceil N/k \rceil$ objects.

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Here, the grades are the “boxes”, of which there are 5. ($k = 5$). Need at least one of the boxes to have at least ten students. ($\lceil N/5 \rceil \geq 10$)
But

$$\lceil N/5 \rceil \geq 10 \text{ exactly when } N/5 > 9.$$  

So the question comes down to finding the least integer $N$ such that $N > 5 \times 9$: $oxed{N = 5 \times 9 + 1}$
The Generalized Pigeonhole Principle says that if $N$ objects are placed into $k$ boxes, then at least one box contains $\lceil N/k \rceil$ objects.

**Example:** How many cards must be selected from a standard deck of 52 cards to guarantee that at least three cards of the same suit appear?
The Generalized Pigeonhole Principle says that if \( N \) objects are placed into \( k \) boxes, then at least one box contains \( \lfloor N/k \rfloor \) objects.

**Example:** How many cards must be selected from a standard deck of 52 cards to guarantee that at least three cards of the same suit appear?

**Answer:**
Here, the suits are the “boxes”, of which there are 4. \((k = 4)\)
The **Generalized Pigeonhole Principle** says that if $N$ objects are placed into $k$ boxes, then at least one box contains $\lceil N/k \rceil$ objects.

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**Example:** How many cards must be selected from a standard deck of 52 cards to guarantee that at least three cards of the same suit appear?

**Answer:**

Here, the suits are the “boxes”, of which there are 4. ($k = 4$)

Need at least one of the boxes to have **at least three** cards.
The **Generalized Pigeonhole Principle** says that if \( N \) objects are placed into \( k \) boxes, then at least one box contains \( \lfloor N/k \rfloor \) objects.

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$$\lfloor N/4 \rfloor \geq 3 \quad \text{exactly when} \quad N/4 > 2.$$
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$$\lceil N/4 \rceil \geq 3 \quad \text{exactly when} \quad N/4 > 2.$$  

So the question comes down to finding the least integer $N$ such that $N > 4 \times 2$: $N = 4 \times 2 + 1$
The **Generalized Pigeonhole Principle** says that if $N$ objects are placed into $k$ boxes, then at least one box contains $\lfloor N/k \rfloor$ objects.

**Non-Example:** How many cards must be selected from a standard deck of 52 cards to guarantee that at least three hearts appear?
The Generalized Pigeonhole Principle says that if \( N \) objects are placed into \( k \) boxes, then at least one box contains \( \lfloor N/k \rfloor \) objects.

**NON-Example:** How many cards must be selected from a standard deck of 52 cards to guarantee that at least three hearts appear?

**Answer:** Since there are 39 cards that are not hearts, we would need to pull at least \( 39 + 3 = 42 \) cards to ensure at least three hearts.
The Generalized Pigeonhole Principle says that if $N$ objects are placed into $k$ boxes, then at least one box contains $\lceil N/k \rceil$ objects.

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**Example:** What is the least number of area codes needed to ensure the availability of at least 25 million distinct phone numbers? (A valid phone number is a sequence of 10 digits, where the first three are the area code, and the 1st and 4th are not 1’s or 0’s.)
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**Ans:** Here, the seven digit phone numbers are the “boxes”
The Generalized Pigeonhole Principle says that if \( N \) objects are placed into \( k \) boxes, then at least one box contains \( \left\lceil \frac{N}{k} \right\rceil \) objects.

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**Ans:** Here, the seven digit phone numbers are the “boxes”, of which there are \( 8 \times 10^6 = 8 \) million. \( (k = 8 \text{ million}) \)
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**Ans:** Here, the seven digit phone numbers are the “boxes”, of which there are $8 \times 10^6 = 8$ million. ($k = 8$ million)

So if we have 25 million phone numbers, at least

$$\lceil 25\text{mil}/8\text{mil} \rceil = 4$$

will have the same 7-digit phone number.
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So if we have 25 million phone numbers, at least

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will have the same 7-digit phone number. Therefore we need 4 area codes.

**You try:** Exercise 23