Math 365 – Wednesday 2/13/19 – Section 5.1: Proof by Induction

Exercise 17. For each of the following, outline a proof by induction. Namely,

- Define $P(n)$.
- Prove the base case.
- To prepare for the inductive step, write down $P(n+1)$.
- Make your inductive hypothesis (fix $n \in \mathbb{N}$ and assume $P(n)$), and prove $P(n+1)$. Be sure to mark where you use the IH (stands for "Induction Hypothesis").
- Make a conclusion.

Where indicated, also write a "final draft" version of your proof.

(a) Show for $n \in \mathbb{N}$ and $r \neq 1$

$$\sum_{i=0}^{n} r^i = \frac{r^{n+1}-1}{r-1}.$$

Also give a final draft proof.

(b) For $n = 1, 2, 3$, calculate $\sum_{i=1}^{n} 2i - 1$ (the sum of the first n odd numbers). Notice that in each case, $\displaystyle\sum_{i=1}^{n} 2i - 1 = n^2$. Show that this is true in general.

(c) Show $n^3 + 2n$ is a multiple of 3 for all $n \in \mathbb{N}$. **Also give a final draft proof.**

(d) Show $n! < n^n$ for $n > 1$.

(e) Suppose $A_1, A_2, \ldots A_N$ and $B_1, B_2, \ldots, B_N$ are sets such that $A_i \subseteq B_i$ for all $1 \leq i \leq N$. Then

$$\bigcup_{i=1}^{N} A_i \subseteq \bigcup_{i=1}^{N} B_i.$$

Also give a final draft proof.

(f) Suppose $A_1, A_2, \ldots A_N$ and B are sets. Then

$$(A_1 - B) \cap (A_2 - B) \cap \cdots \cap (A_N - B) = (A_1 \cap A_2 \cap \cdots \cap A_N) - B.$$

Exercise 18.

(a) Suppose that you know that a golfer plays the first hole of a golf course with an infinite number of holes and that if this golfer plays one hole, then the golfer goes on to play the next hole. Use "proof by induction" reasoning to explain why this golfer must play every hole on the course.

(b) Find the error in the following faulty proof that all horses are the same color.

Proof. Let $P(n)$ be the proposition that all the horses in a set of n horses are the same color. First, $P(1)$ is true since in any group of 1 horse, all the horses must be the same color. Now, fix $n \geq 1$ and assume that $P(n)$ is true, so that all the horses in any set of n horses are the same color. Consider any $n+1$ horses; number these as horses $1, 2, 3, \ldots, n, n+1$. Now the first n of these horses all must have the same color, and the last n of these must also have the same color. Because the set of the first n horses and the set of the last n horses overlap, all $n+1$ must be the same color. Thus $P(n)$ is true for all $n \geq 1$. □

(c) The following is an example of why checking the base case is important.

Let $P(n)$ be the statement $\displaystyle\sum_{i=1}^{n} i = \frac{\left(n+\frac{1}{2}\right)^2}{2}$.

(i) Show that $P(n)$ implies $P(n+1)$. Namely, assume $\sum_{i=1}^{n} i = \frac{\left(n+\frac{1}{2}\right)^2}{2}$ for some $n \geq 1$, and <u>use that to show</u> that $\sum_{i=1}^{n+1} i = \frac{\left((n+1)+\frac{1}{2}\right)^2}{2}$.

(ii) Check the base case, $P(1)$.

(iii) Prove that $P(n)$ is actually <u>false for all $n \in \mathbb{Z}_{\geq 1}$</u>.

Attach at the end of your homework assignment.
Revisiting the handouts "Communicating Mathematics through Homework and Exams" and "Some Guidelines for Good Mathematical Writing", include the following, labeling this as **"Writing exercise"**.

(a) Mark up your finished homework assignment, showing where you followed or failed to follow the mechanical and stylistic issues outlined in the handout *Communicating Mathematics. . . .* This means **treat your write-up as a second-to-last draft**, and go point-by-point through the handout and address instances where you followed or did not follow each direction in your writing. Use a different-colored pen if you have one, and hand in this marked up draft. You do not need to rewrite the result.

   How have you improved this week over your previous homework? How might you improve in the future?

(b) List three or more ways that you succeeded or failed at following the advice in *Some Guidelines. . . .* How have you improved this week over your previous homework? How might you improve in the future?

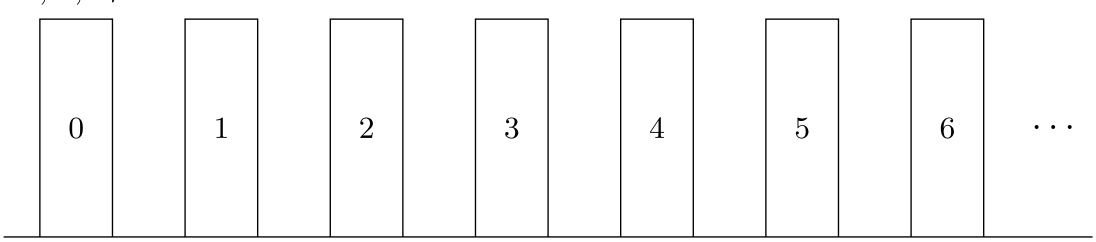
To receive credit for this homework assignment, you must complete this exercise.

Mathematical Induction

Sorites paradox: If $1,000,000$ grains of sand forms a "heap of sand", and removing one grain from a heap leaves it a heap, then a single grain of sand (or even no grains) still forms a heap.

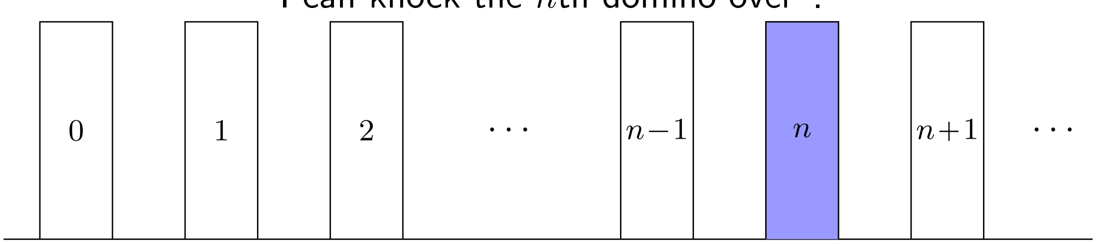
Mathematical induction

Say we have a statement, $P(n)$, that has the natural numbers $n \in \mathbb{Z}_{\geqslant 0}$ as an input.
For example, say you have an infinite row of dominoes, labeled $0, 1, 2, \ldots$:



Let $P(n)$ be the statement

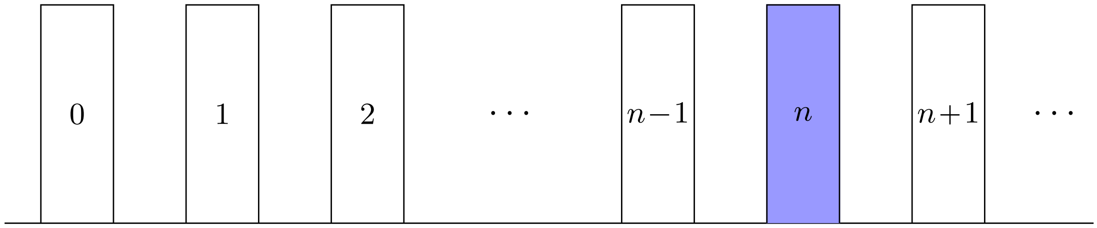
"I can knock the nth domino over".



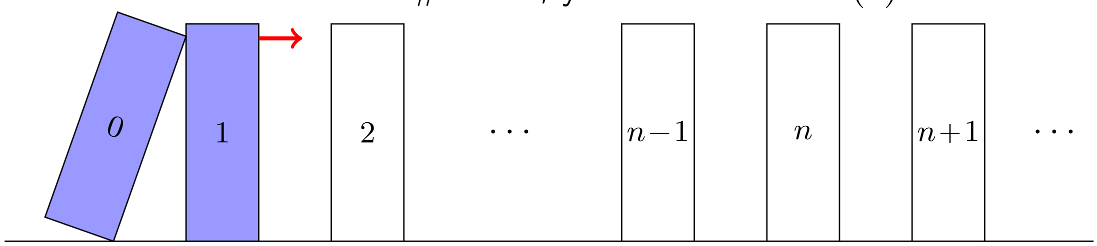
Mathematical induction

Let $P(n)$ be the statement

"I can knock the nth domino over".



Then, if you can show that the 0th domino knocking into the 1st domino with then knock #1 over, you'll show that $P(1)$ is true:



In math: You can show that $P(1)$ is true by proving (a) $P(0)$ is true, and (b) that $P(0)$ implies $P(1)$.

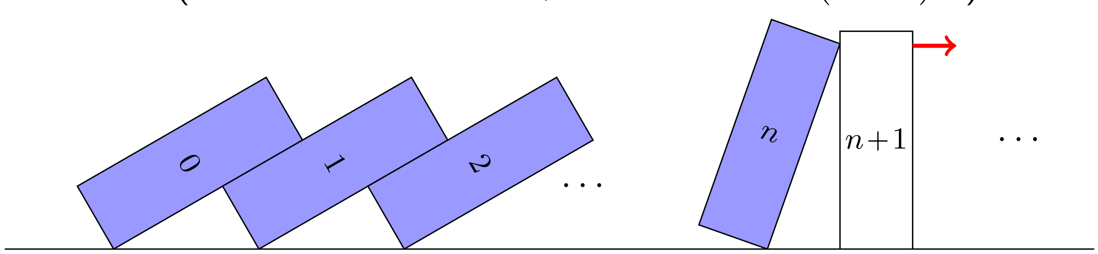
Idea: $P(1)$ will imply $P(2)$, which will imply $P(3)$, and so on. . .

Mathematical induction

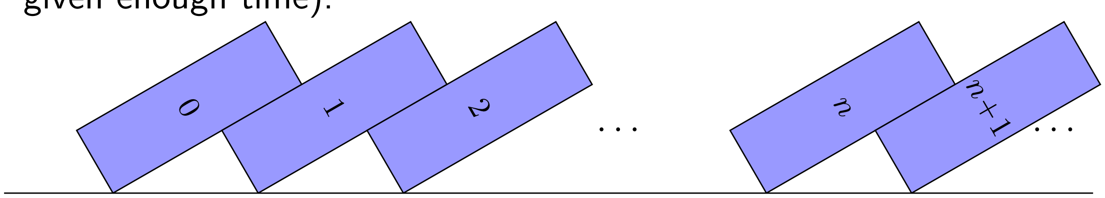
To show that $P(k)$ holds in general, you show that

(a) $P(0)$ is true, and then

(b) for any n, if $P(n)$ is true, then that implies $P(n+1)$ is also true. (If the nth domino falls, then so will the $(n+1)$th)



Then by letting the dominos fall one after the other, eventually each domino will fall (no particular domino will be left standing, given enough time):



Mathematical induction

Theorem: for any $k \in \mathbb{Z}_{\geqslant 0}$, I can knock down the kth domino.

Poof by induction:

First, I can knock down the 0th domino. (“Base case”)

Now, for some $n \in \mathbb{Z}_{\geqslant 0}$, suppose I can knock down the nth domino.

(“Induction hypothesis”)

The nth domino will bump into the $(n+1)$th domino, which will knock it over. So that implies I can knock down the $(n+1)$th domino. (“Induction step”)

Thus, by induction, I can knock down the kth domino for any $k \in \mathbb{Z}_{\geqslant 0}$. $\square$ (“Conclusion”)

Fill in the proof by induction that

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

Proof. *Define* $P(n)$: $P(n)$ is...

Basis case: $P(1)$ is...

Goal: Fix $n \geq 1$, assume $P(n)$, and use it to prove $P(n+1)$, which is...

$P(n+1)$:

Inductive step: Fix $n \geq 1$, assume

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

Then...

Conclusion:

$\square$

Math example: Show $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ by induction.

Proof by induction (final draft). For $n = 1$, we have

$$\sum_{i=1}^{1} i = 1 = \frac{1 * 2}{2},$$

as desired. Now fix $n \geqslant 1$ and assume $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ (for that value of n). Then

$$\begin{aligned}\sum_{i=1}^{n+1} i &= \underbrace{1 + 2 + \cdots + n}_{\sum_{i=1}^{n} i} + (n+1) \\ &= \frac{n(n+1)}{2} + (n+1) \quad \text{(by the inductive hypothesis)} \\ &= \frac{n^2 + n + 2n + 2}{2} = \frac{n^2 + 3n + 2}{2} = \frac{(n+1)(n+2)}{2}.\end{aligned}$$

Thus, the claim holds for all $n \geqslant 1$ by induction. $\square$

Example: Show $n < 2^n$ for all $n \in \mathbb{Z}_{\geqslant 0}$ by induction.

Proof by induction (first draft).

Define $P(n)$: $P(n)$ is "$n < 2^n$".

Base case: The least value of n is 0, so the base case is $P(0)$:

$$0 < 1 = 2^0. \qquad \checkmark$$

Goal: Assume $P(n)$ and show $P(n+1)$, which is

$$P(n+1): \qquad n+1 < 2^{n+1}.$$

Inductive step: (Assume $P(n)$ and show $P(n+1)$)
Fix $n \geqslant 0$ and assume $n < 2^n$ (this is the IH). Then since $n \geqslant 0$,

$$n+1 \overset{\text{IH}}{<} 2^n + 1 \leqslant 2^n + 2^n = 2(2^n) = 2^{n+1}. \qquad \checkmark$$

Conclusion: So since $P(0)$ is true, and $P(n)$ implies $P(n+1)$, we have $P(k)$ is true for all $k \in \mathbb{Z}_{\geqslant 0}$. □

Example: Show $n < 2^n$ for all $n \in \mathbb{Z}_{\geqslant 0}$ by induction.

Proof by induction (final draft).
For $n = 0$, we have

$$0 < 1 = 2^0,$$

as desired. Now, fix $n \geqslant 0$ and assume $n < 2^n$ (for that n). Then since $n \geqslant 0$, we have

$$n+1 < 2^n + 1 \leqslant 2^n + 2^n = 2(2^n) = 2^{n+1}.$$

Thus, the claim holds for all $n \geqslant 0$ by induction. □

Example: Show $n^2 + n$ is even for all $n \in \mathbb{Z}_{\geqslant 0}$ by induction.

Proof by induction (first draft).

Define $P(n)$: $P(n)$ is "$n^2 + n = 2k$ for some integer k".

Base case: (Show $P(0)$) We have

$$0^2 + 0 = 0 = 2 * 0. \qquad \checkmark$$

Goal: Assume $P(n)$ and show $P(n+1)$, which is

$$P(n+1): \qquad (n+1)^2 + (n+1) = 2\ell \text{ for some } \ell \in \mathbb{Z}$$

(Careful!! Don't use the same letter for the IH and $P(n+1)$ since it's *any* integer, not something we get from a formula!!)

Example: Show $n^2 + n$ is even for all $n \in \mathbb{Z}_{\geqslant 0}$ by induction.

Proof by induction (first draft). (Continued from previous slide, where $P(n)$ is "$n^2 + n = 2k$ for some integer k".)

Goal: Assume $P(n)$ and show $P(n+1)$, which is

$$P(n+1): \qquad (n+1)^2 + (n+1) = 2\ell \text{ for some } \ell \in \mathbb{Z}$$

Inductive step: (Assume $P(n)$ and show $P(n+1)$)
Fix $n \geqslant 0$ and assume $n^2 + n = 2k$ for some $k \in \mathbb{Z}$ (this is the IH). Then

$$(n+1)^2 + (n+1) = n^2 + 2n + 1 + n + 1 = \underbrace{(n^2 + n)}_{\text{even by IH}} + (2n+2)$$

$$\overset{\text{IH}}{=} 2k + 2(n+1) = 2\underbrace{(k + n + 1)}_{\in \mathbb{Z}}. \qquad \checkmark$$

Conclusion: So since $P(0)$ is true, and $P(n)$ implies $P(n+1)$, we have $P(k)$ is true for all $k \in \mathbb{Z}_{\geqslant 0}$. $\square$

Example: Show $n^2 + n$ is even for all $n \in \mathbb{Z}_{\geqslant 0}$ by induction.

Proof by induction (final draft). For $n = 0$, we have

$$0^2 + 0 = 0 = 2 * 0,$$

as desired. Next, fix $n \geqslant 0$ and assume $n^2 + n$ is even. Then $n^2 + n = 2k$ for some $k \in \mathbb{Z}$, so that

$$\begin{aligned}(n+1)^2 + (n+1) &= n^2 + 2n + 1 + n + 1 = (n^2 + n) + (2n + 2) \\ &= 2k + 2(n+1) \qquad \text{by the inductive hypothesis,} \\ &= 2(k + n + 1).\end{aligned}$$

So since $k + n + 1 \in \mathbb{Z}$, we have $(n+1)^2 + (n+1)$ is even as well. Thus, the claim holds for all $n \geqslant 0$ by induction. $\square$

Of course, we could have shown this directly!

Example: Show that if $|A| = n$ then $|\mathcal{P}(A)| = 2^n$.

Proof by induction (first draft).

Define $P(n)$: $P(n)$ is "if $|A| = n$ then $|\mathcal{P}(A)| = 2^n$".

Base case: The smallest set is the empty set, so the base case is $P(0)$. In fact, the only set of size 0 is $\varnothing$. So we check $P(0)$ by computing $|\mathcal{P}(\varnothing)|$:

$$|\mathcal{P}(\varnothing)| = |\{\varnothing\}| = 1 = 2^0. \qquad \checkmark$$

Goal: Assume $P(n)$ and show $P(n+1)$, which is

$$P(n+1): \qquad \text{if } |B| = n+1, \text{ then } |\mathcal{P}(B)| = 2^{n+1}.$$

(Careful!! Don't use the same set name for the IH and $P(n+1)$ since they must be different sets!!)

Example: Show that if $|A| = n$ then $|\mathcal{P}(A)| = 2^n$.

Proof by induction (first draft). (Continued from previous slide, where $P(n)$ is "if $|A| = n$ then $|\mathcal{P}(A)| = 2^n$")

Inductive step: (Assume $P(n)$ and show $P(n+1)$)
For any set A of size n, assume $|\mathcal{P}(A)| = 2^n$. Now let B be a set of size $n+1$, and let $b \in B$. Let $A = B - \{b\}$, so that

$$|A| = n \quad \text{and} \quad B = A \cup \{b\}.$$

Then for each subset $X \subseteq A$, there are exactly two subsets of B:

$$X \qquad \text{and} \qquad X \cup \{b\}.$$

So

$$|\mathcal{P}(B)| = 2|\mathcal{P}(A)| \overset{\text{IH}}{=} 2 * 2^n = 2^{n+1}. \quad \checkmark$$

Conclusion: So since $P(0)$ is true, and $P(n)$ implies $P(n+1)$, we have $P(k)$ is true for all $k \in \mathbb{Z}_{\geqslant 0}$. $\square$

Example: Show that if $|A| = n$ then $|\mathcal{P}(A)| = 2^n$.

Proof by induction (final draft). For $n = 0$, we have $A = \varnothing$, and so $\mathcal{P}(A) = \{\varnothing\}$. Thus

$$|\mathcal{P}(\varnothing)| = |\{\varnothing\}| = 1 = 2^0,$$

as desired. Now fix $n \geqslant 0$ and assume for any size-n set A, we have $|\mathcal{P}(A)| = 2^n$. Let B be a set of size $n+1$, and let $b \in B$. Let $A = B - \{b\}$, so that

$$|A| = n \quad \text{and} \quad B = A \cup \{b\}.$$

Then for each subset $X \subseteq A$, there are exactly two subsets of B:

$$X \qquad \text{and} \qquad X \cup \{b\}.$$

So

$$|\mathcal{P}(B)| = 2|\mathcal{P}(A)| = 2 * 2^n = 2^{n+1},$$

by the induction hypothesis. Thus the claim holds for all $n \geqslant 0$ by induction. □

Proof by induction

Outlining your proof:

1. Define $P(n)$.
2. Compute base case.
3. Explicitly state your goal.
4. Do inductive step.
5. State your conclusion.

Rewrite your proof:

1. Write the base case.
2. Fix n and make your inductive hypothesis.
3. Show that the claim holds for $n+1$.
4. State your conclusion.

You try: Exercise 17.