Mathematical Induction

Sorites paradox: If 1,000,000 grains of sand forms a “heap of sand”, and removing one grain from a heap leaves it a heap, then a single grain of sand (or even no grains) still forms a heap.
Mathematical induction

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For example, say you have an infinite row of dominoes, labeled $0, 1, 2, \ldots$:

\[
\begin{array}{cccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & \cdots \\
\end{array}
\]
Mathematical induction

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For example, say you have an infinite row of dominoes, labeled 0, 1, 2, ...:

Let $P(n)$ be the statement "I can knock the $n$th domino over".
Mathematical induction

Let $P(n)$ be the statement

“I can knock the $n$th domino over”.

If you can start by bumping the 0th domino over, that’s showing that $P(0)$ is true:
Mathematical induction

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Then, if you can show that the 0th domino knocking into the 1st domino with then knock #1 over, you’ll show that $P(1)$ is true:
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In math: You can show that $P(1)$ is true by proving (a) $P(0)$ is true, and (b) that $P(0)$ implies $P(1)$.
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In math: You can show that $P(1)$ is true by proving

(a) $P(0)$ is true, and (b) that $P(0)$ implies $P(1)$.

Idea: $P(1)$ will imply $P(2)$, which will imply $P(3)$, and so on...
Mathematical induction

To show that $P(k)$ holds in general, you show that

(a) $P(0)$ is true, and then

(b) for any $n$, if $P(n)$ is true, then that implies $P(n + 1)$ is also true. (If the $n$th domino falls, then so will the $(n + 1)$th)
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Then by letting the dominos fall one after the other, eventually each domino will fall (no particular domino will be left standing, given enough time):
Mathematical induction

**Theorem:** for any $k \in \mathbb{Z}_{\geq 0}$, I can knock down the $k$th domino.
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The \( n \)th domino will bump into the \((n + 1)\)th domino, which will knock it over. So that implies I can knock down the \((n + 1)\)th domino.
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Thus, by induction, I can knock down the $k$th domino for any $k \in \mathbb{Z}_{\geq 0}$. □
Theorem: for any $k \in \mathbb{Z}_{\geq 0}$, I can knock down the $k$th domino.

Proof by induction:
First, I can knock down the 0th domino. ("Base case")

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**Proof by induction:**
First, I can knock down the 0th domino. (**“Base case”**)  

Now, for some $n \in \mathbb{Z}_{\geq 0}$, suppose I can knock down the $n$th domino. (**“Induction hypothesis”**)  

The $n$th domino will bump into the $(n + 1)$th domino, which will knock it over. So that implies I can knock down the $(n + 1)$th domino.  

Thus, by induction, I can knock down the $k$th domino for any $k \in \mathbb{Z}_{\geq 0}$. □
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First, I can knock down the 0th domino. ("Base case")

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The $n$th domino will bump into the $(n + 1)$th domino, which will knock it over. So that implies I can knock down the $(n + 1)$th domino. ("Induction step")

Thus, by induction, I can knock down the $k$th domino for any $k \in \mathbb{Z}_{\geq 0}$. □
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First, I can knock down the 0th domino. \( \text{ (“Base case”) } \)

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Thus, by induction, I can knock down the \( k \)th domino for any \( k \in \mathbb{Z}_{\geq 0} \). \( \square \) \( \text{ (“Conclusion”) } \)
Math example: Show $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ by induction.
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Proof by induction (first draft).

**Define** $P(n)$: $P(n)$ is $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$. 
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$$\sum_{i=1}^{1} i = \frac{1 \times 2}{2}.$$

✓

Goal: Assume $P(n)$ and show $P(n + 1)$, which is

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P(n+1) : \quad \sum_{i=1}^{n+1} i = \frac{(n + 1)((n + 1) + 1)}{2} = \frac{(n + 1)(n + 2)}{2}.
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**Inductive step:** (Assume $P(n)$ and show $P(n + 1)$)

Fix $n \geq 1$ and assume $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ (this is the Inductive Hypothesis, IH).
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\[
\sum_{i=1}^{n+1} i = \sum_{i=1}^{n} i + (n+1)
\]

\[
= \frac{n(n+1)}{2} + (n+1)
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\[
= \frac{n(n+1)+2(n+1)}{2}
\]

\[
= \frac{(n+1)(n+2)}{2}
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Conclusion: So since \( P(1) \) is true, and \( P(n) \) implies \( P(n+1) \), we have \( P(k) \) is true for all \( k \geq 1 \), 2, 3, ...
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\sum_{i=1}^{n+1} i = 1 + 2 + \cdots + n + (n + 1)
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\[
\sum_{i=1}^{n+1} i = 1 + 2 + \cdots + n + (n + 1) = \sum_{i=1}^{n} i + (n + 1) = \frac{n(n+1)}{2} + (n + 1) \quad \text{(by the Inductive Hypothesis)}
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\[
\begin{align*}
\sum_{i=1}^{n+1} i &= 1 + 2 + \cdots + n + (n + 1) \\
&= \frac{n(n+1)}{2} + (n + 1) \\
&= \frac{n^2 + n + 2n + 2}{2} \\
&= \frac{n^2 + 3n + 2}{2}
\end{align*}
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$$\begin{align*}
\sum_{i=1}^{n+1} i &= 1 + 2 + \cdots + n + (n + 1) \\
&= \sum_{i=1}^{n} i + (n + 1) \\
&\overset{\text{IH}}{=} \frac{n(n+1)}{2} + (n + 1) \quad \text{(by the Inductive Hypothesis)} \\
&= \frac{n^2 + n + 2n + 2}{2} = \frac{n^2 + 3n + 2}{2} = \frac{(n + 1)(n + 2)}{2}.
\end{align*}$$

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**Conclusion:** So since $P(1)$ is true, and $P(n)$ implies $P(n + 1)$, we have $P(k)$ is true for all $k = 1, 2, 3, \ldots$. □
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Proof by induction (final draft). For $n = 1$, we have

$$\sum_{i=1}^{1} i = 1 = \frac{1 \times 2}{2},$$

as desired. Now fix $n \geq 1$ and assume $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ (for that value of $n$). Then

$$\sum_{i=1}^{n+1} i = \frac{n(n+1)}{2} + (n + 1) \quad \text{(by the inductive hypothesis)}$$

$$= \frac{n^2 + n + 2n + 2}{2} = \frac{n^2 + 3n + 2}{2} = \frac{(n + 1)(n + 2)}{2}. $$

Thus, the claim holds for all $n \geq 1$ by induction. □
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**Goal:** Assume \( P(n) \) and show \( P(n + 1) \), which is
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P(n + 1) : \quad n + 1 < 2^{n+1}.
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**Inductive step:** (Assume $P(n)$ and show $P(n + 1)$)

Fix $n \geq 0$ and assume $n < 2^n$ (this is the IH).
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Fix $n \geq 0$ and assume $n < 2^n$ (this is the IH). Then since $n \geq 0$,

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\[ \text{IH} \]
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Fix \( n \geq 0 \) and assume \( n < 2^n \) (this is the IH). Then since \( n \geq 0 \),
\[
\text{IH}\quad n + 1 < 2^n + 1 \leq 2^n + 2^n = 2(2^n) = 2^{n+1}. \quad \checkmark
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**Conclusion:** So since \( P(0) \) is true, and \( P(n) \) implies \( P(n + 1) \),
we have \( P(k) \) is true for all \( k \in \mathbb{Z}_{\geq 0} \). \( \square \)
Example: Show $n < 2^n$ for all $n \in \mathbb{Z}_{\geq 0}$ by induction.

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Proof by induction (final draft).
For $n = 0$, we have

$$0 < 1 = 2^0,$$

as desired. Now, fix $n \geq 0$ and assume $n < 2^n$ (for that $n$). Then since $n \geq 0$, we have

$$n + 1 < 2^n + 1 \leq 2^n + 2^n = 2(2^n) = 2^{n+1}.$$

Thus, the claim holds for all $n \geq 0$ by induction. \qed
Example: Show $n^2 + n$ is even for all $n \in \mathbb{Z}_{\geq 0}$ by induction.
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Proof by induction (first draft).

**Define $P(n)$:** $P(n)$ is “$n^2 + n = 2k$ for some integer $k$”.
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Proof by induction (first draft). (Continued from previous slide, where $P(n)$ is “$n^2 + n = 2k$ for some integer $k$”.)

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Fix $n \geq 0$ and assume $n^2 + n = 2k$ for some $k \in \mathbb{Z}$ (this is the IH).
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**Inductive step:** (Assume $P(n)$ and show $P(n+1)$)
Fix $n \geq 0$ and assume $n^2 + n = 2k$ for some $k \in \mathbb{Z}$ (this is the IH). Then

$$(n + 1)^2 + (n + 1) = n^2 + 2n + 1 + n + 1 = \left( n^2 + n \right) + (2n + 2)$$

\[ \text{even by IH} \]

\[ \overset{\text{IH}}{=} 2k + 2(n + 1) = 2(k + n + 1) \]
Example: Show $n^2 + n$ is even for all $n \in \mathbb{Z}_{\geq 0}$ by induction.

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$$= 2k + 2(n + 1) = 2(k + n + 1) \in \mathbb{Z}.$$ 

**Conclusion:** So since $P(0)$ is true, and $P(n)$ implies $P(n + 1)$, we have $P(k)$ is true for all $k \in \mathbb{Z}_{\geq 0}$. □
Example: Show $n^2 + n$ is even for all $n \in \mathbb{Z}_{\geq 0}$ by induction.

Proof by induction (final draft). For $n = 0$, we have

$$0^2 + 0 = 0 = 2 \times 0,$$

as desired. Next, fix $n \geq 0$ and assume $n^2 + n$ is even. Then $n^2 + n = 2k$ for some $k \in \mathbb{Z}$, so that

$$(n + 1)^2 + (n + 1) = n^2 + 2n + 1 + n + 1 = (n^2 + n) + (2n + 2)$$

$$= 2k + 2(n + 1) \quad \text{by the inductive hypothesis},$$

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So since $k + n + 1 \in \mathbb{Z}$, we have $(n + 1)^2 + (n + 1)$ is even as well. Thus, the claim holds for all $n \geq 0$ by induction. \qed
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as desired. Next, fix \( n \geq 0 \) and assume \( n^2 + n \) is even. Then \( n^2 + n = 2k \) for some \( k \in \mathbb{Z} \), so that
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So since \( k + n + 1 \in \mathbb{Z} \), we have \( (n + 1)^2 + (n + 1) \) is even as well. Thus, the claim holds for all \( n \geq 0 \) by induction. \( \square \)

Of course, we could have shown this directly!
Example: Show that if $|A| = n$ then $|\mathcal{P}(A)| = 2^n$. 

Proof by induction (first draft).

Define $P_{p_nq}$: $P_{p_nq}$ is "if $|A| = n$ then $|\mathcal{P}(A)| = 2^n$.

Base case: The smallest set is the empty set, so the base case is $P_{p_0q}$.

In fact, the only set of size $0$ is $\emptyset$. So we check $P_{p_0q}$ by computing $|\mathcal{P}(\emptyset)|$: $|\emptyset| = 1 = 2^0$.

Goal: Assume $P_{p_nq}$ and show $P_{p_{n+1}q}$, which is $P_{p_{n+1}q}$: if $|B| = n+1$, then $|\mathcal{P}(B)| = 2^{n+1}$.

(Careful!! Don't use the same set name for the IH and $P_{p_{n+1}q}$ since they must be different sets!!)
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$$P(n + 1): \text{ if } |B| = n + 1, \text{ then } |\mathcal{P}(B)| = 2^{n+1}.$$ (Careful!! Don’t use the same set name for the IH and $P(n + 1)$ since they must be different sets!!)
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**Inductive step:** (Assume $P(n)$ and show $P(n + 1)$)

For any set $A$ of size $n$, assume $|\mathcal{P}(A)| = 2^n$. 
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Then for each subset \(X \subseteq A\), there are exactly two subsets of \(B\):

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\(|\mathcal{P}(B)| = 2|\mathcal{P}(A)| \overset{\text{IH}}{=} 2 * 2^n = 2^{n+1}. \quad \checkmark\)
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Then for each subset $X \subseteq A$, there are exactly two subsets of $B$: $X$ and $X \cup \{b\}$.

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**Conclusion:** So since $P(0)$ is true, and $P(n)$ implies $P(n + 1)$, we have $P(k)$ is true for all $k \in \mathbb{Z}_{\geq 0}$.  \(\square\)
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Proof by induction (final draft). For $n = 0$, we have $A = \emptyset$, and so $\mathcal{P}(A) = \{\emptyset\}$. Thus

$$|\mathcal{P}(\emptyset)| = |\{\emptyset\}| = 1 = 2^0,$$

as desired. Now fix $n \geq 0$ and assume for any size-$n$ set $A$, we have $|\mathcal{P}(A)| = 2^n$. Let $B$ be a set of size $n + 1$, and let $b \in B$. Let $A = B - \{b\}$, so that

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Then for each subset $X \subseteq A$, there are exactly two subsets of $B$: $X$ and $X \cup \{b\}$.

So

$$|\mathcal{P}(B)| = 2|\mathcal{P}(A)| = 2 \cdot 2^n = 2^{n+1},$$

by the induction hypothesis. Thus the claim holds for all $n \geq 0$ by induction. \qed
Proof by induction

Outlining your proof:

1. Define $P(n)$.
2. Compute base case.
3. Explicitly state your goal.
4. Do inductive step.
5. State your conclusion.

You try: Exercise 17.
Proof by induction

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Rewrite your proof:
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