Functions

Some functions you might be familiar with:

\[ f(x) = x^2, \quad f(x) = 3x - 2, \quad f(x) = \sqrt{x}, \quad f(x, y) = \left( \begin{array}{c} x \\ y \end{array} \right). \]
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A couple more we’ll need:
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- For \( x \in \mathbb{R} \), the **floor** of \( x \) is the greatest integer that is less than or equal to \( x \), written \( \lfloor x \rfloor \).
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- For \( x \in \mathbb{R} \), the **floor** of \( x \) is the greatest integer that is less than or equal to \( x \), written \([x]\). For example,

\[ [1/2] = 0, \quad [-1/2] = -1, \quad [13] = 13, \quad [\pi] = 3. \]
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  \[
  \]

- For \( x \in \mathbb{R} \), the **ceiling** of \( x \) is the least integer that is greater than or equal to \( x \), written \([x]\).
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A couple more we’ll need:

- For \( x \in \mathbb{R} \), the floor of \( x \) is the greatest integer that is less than or equal to \( x \), written \([x]\). For example,
  \[
  \]

- For \( x \in \mathbb{R} \), the ceiling of \( x \) is the least integer that is greater than or equal to \( x \), written \([x]\). For example,
  \[
  \]
Functions

- For $x \in \mathbb{R}$, the floor of $x$ is the greatest integer that is less than or equal to $x$, written $\lfloor x \rfloor$. For example,

$$\lfloor 1/2 \rfloor = 0, \quad \lfloor -1/2 \rfloor = -1, \quad \lfloor 13 \rfloor = 13, \quad \lfloor \pi \rfloor = 3.$$

- For $x \in \mathbb{R}$, the ceiling of $x$ is the least integer that is greater than or equal to $x$, written $\lceil x \rceil$. For example,

$$\lceil 1/2 \rceil = 1, \quad \lceil -1/2 \rceil = 0, \quad \lceil 13 \rceil = 13, \quad \lceil \pi \rceil = 4.$$

- The absolute value of a real number $x$ is

$$|x| = \begin{cases} x & \text{if } x \text{ is nonnegative}, \\ -x & \text{if } x \text{ is negative}, \end{cases}$$

so that $|x|$ is always nonnegative.
Functions

- For $x \in \mathbb{R}$, the **floor** of $x$ is the greatest integer that is less than or equal to $x$, written $[x]$. For example,


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  so that $|x|$ is always nonnegative. For example,

  $|1/2| = 1/2, \quad |-1/2| = 1/2, \quad |0| = 0, \quad |\pi| = \pi.$
What makes a function?

- You need a domain (input).

The function should be well-defined (part 1): for every input, there is exactly one output. Namely, if \( f(a) = b_1 \) and \( f(a) = b_2 \), then \( b_1 = b_2 \).

The domain together with a function determines a range or image (output).

Example Consider \( f(x) = x^2 \). If the domain is \( \mathbb{R} \), then the range is \( \mathbb{R} \geq 0 \). If the domain is \( \mathbb{R} \geq 1 \), then the range is \( \mathbb{R} \geq 1 \). Either way, \( f \) is well-defined "on its domain".
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What makes a function?

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  \]

\[b_1\] 
\[?\] 
\[b_2\] 

\[a\] 

Bad: 
\[\sqrt{4} \rightarrow -2, \sqrt{4} \rightarrow 2\]
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**Example**

Consider \( f(x) = x^2 \).

If the domain is \( \mathbb{R} \), then the range is \( \mathbb{R}_{\geq 0} \).
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If the domain is \( \{-1\} \)
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Like we can pick a universal set, we can also pick a codomain, a set containing the range of $f$. 
Like we can pick a universal set, we can also pick a **codomain**, a set containing the range of $f$.

If $f$ is a function with domain $A$ and codomain $B$, we say $f$ is a **function** or **map** or **transformation** from $A$ to $B$, and we write

$$f : A \to B.$$
Like we can pick a universal set, we can also pick a codomain, a set containing the range of $f$.
If $f$ is a function with domain $A$ and codomain $B$, we say $f$ is a function or map or transformation from $A$ to $B$, and we write
$$f : A \rightarrow B.$$ 
For $a \in A$, we write
$$f : a \mapsto f(a),$$
where “$\mapsto$” reads “maps to”.

If you have a function $f : A \rightarrow B$, and $A_1 \subseteq A$, you can restrict $f$ to the domain $A_1$, written $f|_{A_1} : A_1 \rightarrow B$.
This means that the definition of the function doesn't change, you just consider its image on fewer elements.
If you pick a bad codomain, your expression is no longer a function (not well-defined, part 2).
Example $f : \mathbb{R} \rightarrow \mathbb{Z}$ defined by $x \mapsto x$ is not a function.
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If you have a function \( f : A \rightarrow B \), and \( A' \subseteq A \), you can restrict \( f \) to the domain \( A' \), written

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f|_{A'} : A' \rightarrow B.
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If you have a function $f : A \rightarrow B$, and $A' \subseteq A$, you can **restrict** $f$ to the domain $A'$, written

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$$f : \mathbb{R} \rightarrow \mathbb{Z}$$

defined by $x \mapsto x$

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\[ f : \mathbb{R} \rightarrow \mathbb{R} \]

\[ x \mapsto x^2. \]

Then the image of \( f \) is \( \mathbb{R}_{\geq 0} \).
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Consider the function

\[ f : \mathbb{R} \to \mathbb{R} \]
\[ x \mapsto x^2. \]

Then the image of \( f \) is \( \mathbb{R}_{\geq 0} \). If we restrict \( f \) to \( \{ -1 \} \subseteq \mathbb{R} \), the image of \( f|_{\{-1\}} : \{-1\} \to \mathbb{R} \) is \( \{1\} \).
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Then the image of \( f \) is \( \mathbb{R}_{\geq 0} \). If we restrict \( f \) to \( \{-1\} \subseteq \mathbb{R} \), the image of \( f|_{\{-1\}} : \{-1\} \rightarrow \mathbb{R} \) is \( \{1\} \).

The functions

\[ g : \mathbb{R} \rightarrow \mathbb{C} \quad \text{and} \quad h : \mathbb{R} \rightarrow \mathbb{C} \cup \mathbb{C}^{15} \]
\[ x \mapsto x^2 \quad \text{and} \quad x \mapsto x^2 \]

both have image \( \mathbb{R}_{\geq 0} \).
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The functions

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The map

\[ \varphi : \mathbb{R} \to \mathbb{Z} \]
\[ x \mapsto x^2 \]

is not well-defined, since the image is not contained in the codomain.
The image of an element \( a \in A \) is just \( f(a) \).
The image of an element \( a \in A \) is just \( f(a) \). The preimage is defined on any element of subset of the codomain. Namely, the preimage of \( b \in B \) is the set of elements \( a \in A \) such that \( f(a) = b \):

\[
    f^{-1}(b) = \{ a \in A \mid f(a) = b \}.
\]

Notice, either way, a preimage is a set!!
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The preimage of a subset \( B' \subseteq B \) is defined similarly, only using containment:

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f^{-1}(B') = \{ a \in A \mid f(a) \in B' \}.
\]

Notice, either way, a preimage is a set!! A function \( f : A \rightarrow B \) is invertible if for every \( b \in B \), \( f^{-1}(b) \) has exactly one element.
A function is called **one-to-one** or **injective** if every element in the range has at most one element in its preimage.
A function is called **one-to-one** or **injective** if every element in the range has at most one element in its preimage. Some examples of **injective functions**:

\[ f(x) = 3x - 5 \text{ with domain } \mathbb{C} \]
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Some examples of **injective functions**:

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A function is called one-to-one or injective if every element in the range has at most one element in its preimage. Some examples of injective functions:

\[ f(x) = 3x - 5 \text{ with domain } \mathbb{C}, \quad f(x) = x^2 \text{ with domain } \mathbb{R}_{\geq 0}, \]

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\[ A \rightarrow f \rightarrow B \]
A function is called **one-to-one** or **injective** if every element in the range has at most one element in its preimage.

Some examples of **functions that are not injective**: 

\[ f(x) = 3x - 5 \] with domain time on a clock
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Some examples of functions that are **not** injective:

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A function is called **one-to-one** or **injective** if every element in the range has at most one element in its preimage.

Some examples of **functions that are not injective**:

\[
\begin{align*}
f(x) &= 3x - 5 \text{ with domain time on a clock}, \\
f(x) &= x^2 \text{ with domain } \mathbb{R}, \\
f(x) &= \lfloor x \rfloor \text{ with domain } \mathbb{Q}
\end{align*}
\]
A function is called **one-to-one** or **injective** if every element in the range has at most one element in its preimage. Some examples of **functions that are not injective:**

\[ f(x) = 3x - 5 \] with domain time on a clock,
\[ f(x) = x^2 \] with domain \( \mathbb{R} \),
\[ f(x) = \lfloor x \rfloor \] with domain \( \mathbb{Q} \),
A function is called *onto* or *surjective* if the codomain and the image are the same thing.

Some examples of *surjective functions*:

\[ f(x) = 3x - 5 \] with domain and codomain \( \mathbb{C} \),
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Some examples of surjective functions:

- \( f(x) = 3x - 5 \) with domain and codomain \( \mathbb{C} \),
- \( f(x) = x^2 \) with domain \( \mathbb{R} \) and codomain \( \mathbb{R}_{\geq 0} \),
A function is called onto or surjective if the codomain and the image are the same thing.

Some examples of surjective functions:

\[ f(x) = 3x - 5 \] with domain and codomain \( \mathbb{C} \),
\[ f(x) = x^2 \] with domain \( \mathbb{R} \) and codomain \( \mathbb{R}_{\geq 0} \),
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Some examples of surjective functions:

\[ f(x) = 3x - 5 \text{ with domain and codomain } \mathbb{C}, \]
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\[ f(x) = 3x - 5 \] with domain \( \mathbb{R} \) and codomain \( \mathbb{C} \)
A function is called \textit{onto} or \textit{surjective} if the codomain and the image are the same thing.

Some examples of \textbf{functions that are not surjective}:

\[ f(x) = 3x - 5 \] with domain \( \mathbb{R} \) and codomain \( \mathbb{C} \),
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Some examples of functions that are **not** surjective:

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- \( f(x) = \lfloor x \rfloor \) with domain and codomain \( \mathbb{Q} \)
A function is called onto or surjective if the codomain and the image are the same thing. Some examples of functions that are not surjective:

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A function that is both injective and surjective is bijective or a one-to-one correspondence.
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\[ A \rightarrow B \]

**No:**
\[ \begin{array}{ccc}
\bullet c & \rightarrow & y \\
\bullet b & \rightarrow & x \\
\bullet a & \rightarrow & B \\
A \end{array} \]

**No:**
\[ \begin{array}{ccc}
\bullet c & \rightarrow & 3 \\
\bullet b & \rightarrow & 2 \\
\bullet a & \rightarrow & 1 \\
A & \rightarrow & B \\
\end{array} \]

**Yes:**
\[ \begin{array}{ccc}
\bullet c & \rightarrow & 3 \\
\bullet b & \rightarrow & 2 \\
\bullet a & \rightarrow & 1 \\
A & \rightarrow & B \\
\end{array} \]

**Theorem**

A function \( f : A \rightarrow B \) is bijective if and only if it is invertible.
A function that is both injective and surjective is **bijective** or a one-to-one correspondence.

Theorem

* A function $f : A \rightarrow B$ is bijective if and only if it is invertible.

You try: Exercise 8.
Let \( f : A \to B \) and \( g : B \to C \).

Then the composition of \( g \) and \( f \) is

\[
g \circ f = g(f(a)) : A \to C.
\]
Let \( f : A \rightarrow B \) and \( g : B \rightarrow C \).

Then the composition of \( g \) and \( f \) is

\[
g \circ f = g(f(a)) : A \rightarrow C.
\]

**Example**

Let

\[
\begin{align*}
A & \rightarrow B \\
\begin{array}{c}
a \\
b \\
c
\end{array} & \rightarrow \begin{array}{c}
0 \\
1 \\
2 \\
3
\end{array}
\end{align*}
\]

and

\[
\begin{align*}
B & \rightarrow C \\
\begin{array}{c}
0 \\
1 \\
2 \\
3
\end{array} & \rightarrow \begin{array}{c}
x \\
y \\
z
\end{array}
\end{align*}
\]

What is \( g \circ f \)?
Let
\[ f : A \to B \quad \text{and} \quad g : B \to C. \]

Then the \textbf{composition} of \( g \) and \( f \) is
\[ g \circ f = g(f(a)) : A \to C. \]

\textbf{Example}
Let \( f(x) = x^2 + 1 \) and let \( g(x) = \lfloor x \rfloor \), both with domain and codomain \( \mathbb{R} \).
Let \( f : A \to B \) and \( g : B \to C \).

Then the **composition** of \( g \) and \( f \) is

\[
g \circ f = g(f(a)) : A \to C.
\]

**Example**

Let \( f(x) = x^2 + 1 \) and let \( g(x) = \lfloor x \rfloor \), both with domain and codomain \( \mathbb{R} \). Since the domain and codomain are equal for both, I can consider both \( f \circ g \) and \( g \circ f \).
Let 

\[ f : A \to B \quad \text{and} \quad g : B \to C. \]

Then the composition of \( g \) and \( f \) is

\[ g \circ f = g(f(a)) : A \to C. \]

**Example**

Let \( f(x) = x^2 + 1 \) and let \( g(x) = \lfloor x \rfloor \), both with domain and codomain \( \mathbb{R} \). Since the domain and codomain are equal for both, I can consider both \( f \circ g \) and \( g \circ f \). We have

\[ f \circ g = \lfloor x \rfloor^2 + 1 \quad \text{and} \quad g \circ f = \lfloor x^2 + 1 \rfloor. \]
Let
\[ f : A \to B \quad \text{and} \quad g : B \to C. \]

Then the composition of \( g \) and \( f \) is
\[ g \circ f = g(f(a)) : A \to C. \]

**Example**

Let \( f(x) = x^2 + 1 \) and let \( g(x) = [x] \), both with domain and codomain \( \mathbb{R} \). Since the domain and codomain are equal for both, I can consider both \( f \circ g \) and \( g \circ f \). We have
\[ f \circ g = [x]^2 + 1 \quad \text{and} \quad g \circ f = [x^2 + 1]. \]

**You try:** Exercise 9.
Let 

\[ f : A \to B \quad \text{and} \quad g : B \to C. \]

Then the composition of \( g \) and \( f \) is

\[ g \circ f = g(f(a)) : A \to C. \]

**Example**

Let \( f(x) = x^2 + 1 \) and let \( g(x) = [x] \), both with domain and codomain \( \mathbb{R} \). Since the domain and codomain are equal for both, I can consider both \( f \circ g \) and \( g \circ f \). We have

\[ f \circ g = [x]^2 + 1 \quad \text{and} \quad g \circ f = [x^2 + 1]. \]

You try: Exercise 9.

**Theorem**

Let \( f : A \to B \) and \( g : B \to C \) be functions. If both \( f \) and \( g \) are one-to-one functions, then \( g \circ f \) is also one-to-one.
Let \( f : A \rightarrow B \) and \( g : B \rightarrow C \).

Then the composition of \( g \) and \( f \) is

\[
g \circ f = g(f(a)) : A \rightarrow C.
\]

**Example**

Let \( f(x) = x^2 + 1 \) and let \( g(x) = \lfloor x \rfloor \), both with domain and codomain \( \mathbb{R} \). Since the domain and codomain are equal for both, I can consider both \( f \circ g \) and \( g \circ f \). We have

\[
f \circ g = \lfloor x \rfloor^2 + 1 \quad \text{and} \quad g \circ f = \lfloor x^2 + 1 \rfloor.
\]

You try: Exercise 9.

**Theorem**

Let \( f : A \rightarrow B \) and \( g : B \rightarrow C \) be functions. If both \( f \) and \( g \) are one-to-one functions, then \( g \circ f \) is also one-to-one.

You try: Exercise 10.