Warm up

Recall that the power set of a set A is

$$\mathcal{P} = \{X \mid X \subseteq A\}$$

.

1. What is $|\varnothing|$? What is $|\{\varnothing\}|$?
2. Let $A = \{x\}$. Calculate $\mathcal{P}(A)$ and $\mathcal{P}(\mathcal{P}(A))$.
3. Let $A = \varnothing$. Calculate $\mathcal{P}(A)$ and $\mathcal{P}(\mathcal{P}(A))$.
4. Give an example of a set A such that $A \cap \mathcal{P}(A) = \varnothing$.
5. Give an example of a set A such that $A \cap \mathcal{P}(A) \neq \varnothing$.
6. True or false and why: For any set A, $\{\varnothing\} \subseteq \mathcal{P}(\mathcal{P}(A))$.
7. Explain why $\mathcal{P}(A) \cap \mathcal{P}(\mathcal{P}(A)) \neq \varnothing$.

Some shorthands you'll see in the book:

symbol:	means:	example:
$\in$	"in", "contained in"	"$x \in \mathbb{R}$" means "x is a real number".
$\forall$	"for all"	$A \subseteq B$ if $\forall a \in A$, we have $a \in B$.
$\wedge$	"and"	$A \cap B = \{x \in U \mid (x \in A) \wedge (x \in B)\}$.
$\vee$	"or" (inclusive)	$A \cup B = \{x \in U \mid (x \in A) \vee (x \in B)\}$.
$\neg$	"not"	$\overline{A} = \{x \in U \mid \neg(x \in A)\}$.

Put a priority on clarity!

Writing mathematics is not that different that any other writing. In journalism, clear and articulate writing is as important as content; the same is true in math. Don't make your reader work too hard to understand what you're trying to convey! In short, use symbols sparingly–go for clarity, not just saving space.

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Draw the universal set at a rectangle.
Inside that rectangle, indicate a set by drawing a closed loop (usually a circle, but not always) where the object in the set are the points inside that closed loop.

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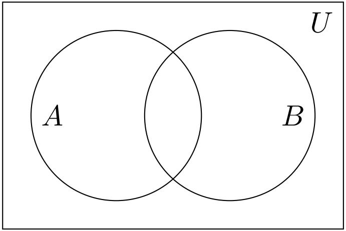

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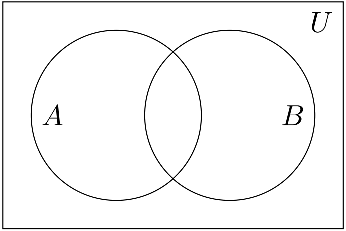

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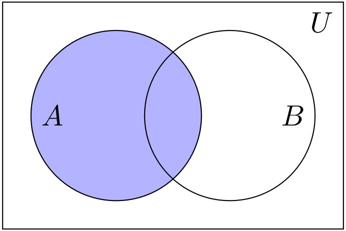
A looks like 

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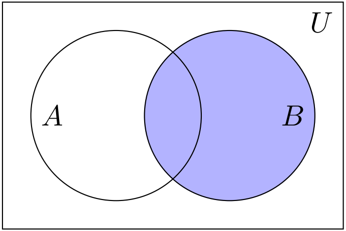

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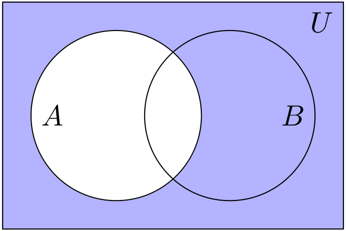

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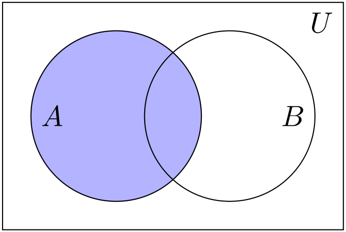

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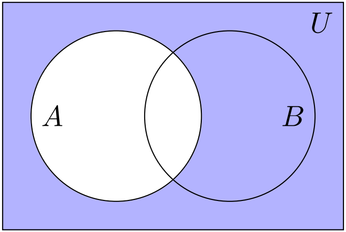

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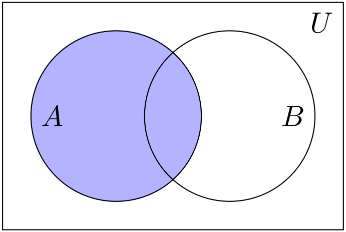

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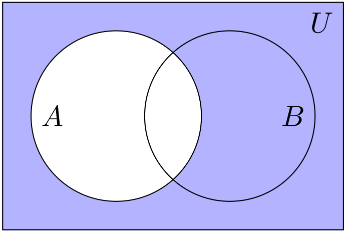

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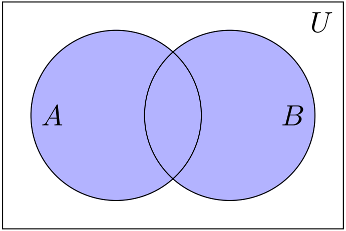

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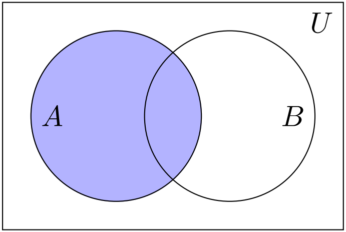

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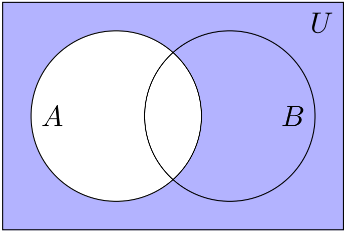

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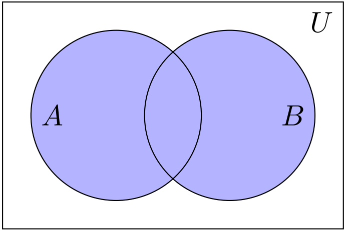

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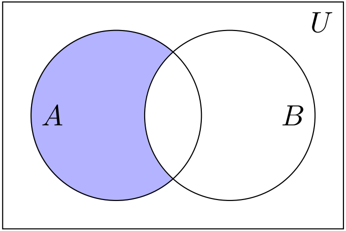

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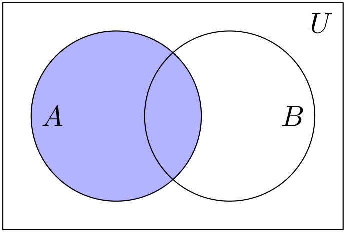

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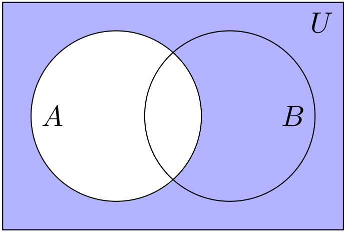

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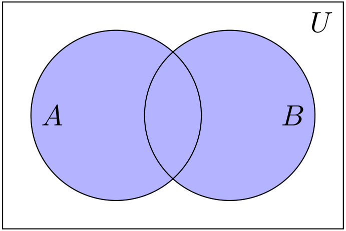

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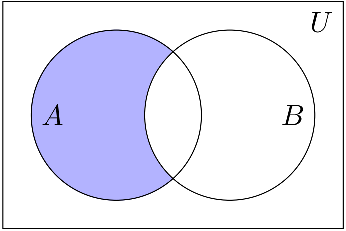

You try: Do Exercise 3

Infinite unions and intersections

Recall summation and product notation

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n, \qquad \sum_{i=1}^{\infty} a_i = a_1 + a_2 + \cdots \ ,$$

$$\prod_{i=1}^{n} a_i = a_1 \cdot a_2 \cdots a_n, \qquad \prod_{i=1}^{\infty} a_i = a_1 \cdot a_2 \cdots \ ,$$

for numbers $a_1, a_2, \ldots$.

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For example,

$$A_1 = \{1\}, \quad A_2 = \{1, 2\}, \quad \ldots \qquad A_i = \{1, 2, 3, \ldots, i\}.$$

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$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n, \qquad \bigcup_{i=1}^{\infty} A_i = A_1 \cup A_2 \cup \cdots$$

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \cdots \cap A_n, \qquad \bigcap_{i=1}^{\infty} A_i = A_1 \cap A_2 \cap \cdots .$$

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You try: Exercise 4.

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- $W \neq X$ because $3 \in X$ but $3 \notin W$ ($X \nsubseteq W$).
- $W \neq Y$ because $2 \in W$ but $2 \notin Y$ ($W \nsubseteq Y$).
- $W = Z$ because

  $1 \in W$ and $1 \in Z$;

  $2 \in W$ and $2 \in Z$;

  and there are no other elements in W or Z ($W \subseteq Z$ and $Z \subseteq W$).

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(if $a \in A$, then $a \in B$ too)

and

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the "and vice versa" part.

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Example

If $A_i = \{1, 2, \ldots, i\}$ for $i = 1, 2, 3, \ldots$, then $\bigcup_{i=1}^{\infty} A_i = \mathbb{Z}_{>0}$.

Proof.

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Therefore $\mathcal{A} = \mathbb{Z}_{>0}$. $\square$

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Proof.

Let $\mathcal{A} = \bigcup_{i=1}^{\infty} A_i$.

First, if $a \in \mathcal{A}$, then a is an element of one of the A_i's. But since $A_i \subseteq \mathbb{Z}_{>0}$ for all $i = 1, 2, 3, \ldots$, we get that $a \in \mathbb{Z}_{>0}$. Thus $\mathcal{A} \subseteq \mathbb{Z}_{>0}$.

Next, if $i \in \mathbb{Z}_{>0}$, then i is an element of A_i, which is a subset of $\mathcal{A}$. Therefore i is and element of $\mathcal{A}$. So $\mathbb{Z}_{>0}$ is a subset of $\mathcal{A}$.

Therefore $\mathcal{A} = \mathbb{Z}_{>0}$. □

You try: Do Exercise 5.

Let A, B, C be sets contained in a universal set U.
The following identities are our core set operations.

Identity	Name
$A \cap U = A \cup \varnothing = A$	Identity laws
$A \cup U = U$ and $A \cap \varnothing = \varnothing$	Domination laws
$A \cup A = A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
$\overline{A \cup B} = \overline{A} \cap \overline{B}$ (Exercise 6) $\overline{A \cap B} = \overline{A} \cup \overline{B}$	**De Morgan's laws**
$A \cup (A \cap B) = A$ and $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ and $A \cap \overline{A} = \varnothing$	Complement laws