From last time:

A Pythagorean triple is a triplet of positive integers \( a, b, c \in \mathbb{Z}_{>0} \) satisfying \( a^2 + b^2 = c^2 \).

Ex: \( 3^2 + 4^2 = 5^2 \), \( 5^2 + 12^2 = 13^2 \), and \( 8^2 + 15^2 = 17^2 \).

Last time, we used factorization and divisors to help us prove the following.

1. If \( (a, b, c) \) is a Pythagorean triple, then so is \( (na, nb, nc) \) for any \( n \in \mathbb{Z}_{>0} \).
2. All primitive Pythagorean triples (those with no common divisors) are characterized by

\[
\begin{align*}
    a &= st, \\
    b &= \frac{s^2 - t^2}{2}, \\
    c &= \frac{s^2 + t^2}{2},
\end{align*}
\]

for odd integers \( s > t \geq 1 \) with no common factors.

Today: Another approach, using geometry.

Pythagorean triples and the unit circle

For any \( c \neq 0 \), we have

\( (a, b, c) \) is a solution to \( a^2 + b^2 = c^2 \)

if and only if

\( (a, b, c) \) is a solution to \( \left( \frac{a}{c} \right)^2 + \left( \frac{b}{c} \right)^2 = 1 \).

Let \( x = a/c \) and \( y = b/c \). Then solutions look like

\[
\begin{align*}
x^2 + y^2 &= 1: \\
(x, y)
\end{align*}
\]

Integer solutions \( (a, b, c) \) occur whenever \( x \) and \( y \) are rational.
Pythagorean triples and the unit circle

\[ x^2 + y^2 = 1 \]

Integer solutions to \( a^2 + b^2 = c^2 \) occur whenever \( x \) and \( y \) are rational. (Let \( c \) be any common multiple of the denominators of \( x \) and \( y \).)

**Four obvious rational points:** \((1,0), (0,1), (-1,0), \) and \((0, -1)\).

Take, for example, the point \( P = (-1,0) \).

Now let \((q, r)\) be any other rational point \((q, r) \in \mathbb{Q} \) on the circle. Consider the line \( L \) connecting those two points. Rational slope!
Pythagorean triples and the unit circle

If we take a line through $P = (-1, 0)$ and another rational point $(q, r)$ on the unit circle, that line will have rational slope.

$x^2 + y^2 = 1$

Conversely, take any line with rational slope $m$ that intersects $P$,

$L : y = m(x + 1), \quad m \in \mathbb{Q}$

(\text{using point-slope formula}).

Let $(x, y)$ be the other point where the line intersects the circle. Solve.
Pythagorean triples and the unit circle

\[ x^2 + y^2 = 1 \]

\[ L : y = m(x + 1), \quad m \in \mathbb{Q} \]

Two points of intersection:
\((-1, 0)\) and \(\left(\frac{1-m^2}{1+m^2}, \frac{2m}{1+m^2}\right)\)

Both rational!!

**Theorem**

*Every point on the circle* \(x^2 + y^2 = 1\) *whose coordinates are rational numbers can be obtained from the formula*

\[(x, y) = \left(\frac{1-m^2}{1+m^2}, \frac{2m}{1+m^2}\right)\]

*by substituting in rational numbers for* \(m\) *or taking the limit* \(m \to \infty\).
Theorem

Every point on the circle \( x^2 + y^2 = 1 \) whose coordinates are rational numbers can be obtained from the formula

\[
(x, y) = \left( \frac{1 - m^2}{1 + m^2}, \frac{2m}{1 + m^2} \right)
\]

by substituting in rational numbers for \( m \) or taking the limit \( m \to \infty \).

Relating back to last time: rational points \((x, y)\) on the unit circle correspond to primitive Pythagorean triples \((a, b, c)\) as follows:

Process:

- Put \( x \) and \( y \) into lowest terms.
- Let \( c \) be the smallest common multiple of their denominators.
- Let \( a = xc \) and \( b = yc \)

Example:

\[
(x, y) = (3/5, 4/5)
\]

\[
c = 5
\]

\[
a = 3 \text{ and } b = 4
\]

Last time: \((a, b, c) = (st, \frac{1}{2}(s^2 - t^2), \frac{1}{2}(s^2 + t^2))\)

Substitute \( m = u/v \). Then let \( u = \frac{1}{2}(s + t), v = \frac{1}{2}(s - t) \).