In this section we give geometric definitions of parabolas, ellipses, and hyperbolas and derive their standard equations. They are called conic sections, or conics, because they result from intersecting a cone with a plane as shown in Figure 1.

**Figure 1**

Conics

**Parabolas**

A parabola is the set of points in a plane that are equidistant from a fixed point $F$ (called the focus) and a fixed line (called the directrix). This definition is illustrated by Figure 2. Notice that the point halfway between the focus and the directrix lies on the parabola; it is called the vertex. The line through the focus perpendicular to the directrix is called the axis of the parabola.

In the 16th century Galileo showed that the path of a projectile that is shot into the air at an angle to the ground is a parabola. Since then, parabolic shapes have been used in designing automobile headlights, reflecting telescopes, and suspension bridges. (See Challenge Problem 2.14 for the reflection property of parabolas that makes them so useful.)

We obtain a particularly simple equation for a parabola if we place its vertex at the origin $O$ and its directrix parallel to the $x$-axis as in Figure 3. If the focus is the point $(0, p)$, then the directrix has the equation $y = -p$. If $P(x, y)$ is any point on the parabola, then the distance from $P$ to the focus is

$$|PF| = \sqrt{x^2 + (y - p)^2}$$

and the distance from $P$ to the directrix is $|y + p|$. (Figure 3 illustrates the case where $p > 0$.) The defining property of a parabola is that these distances are equal:

$$\sqrt{x^2 + (y - p)^2} = |y + p|$$

We get an equivalent equation by squaring and simplifying:

$$x^2 + (y - p)^2 = |y + p|^2 = (y + p)^2$$

$$x^2 + y^2 - 2py + p^2 = y^2 + 2py + p^2$$

$$x^2 = 4py$$

An equation of the parabola with focus $(0, p)$ and directrix $y = -p$ is

$$x^2 = 4py$$
If we write $a = 1/(4p)$, then the standard equation of a parabola (1) becomes $y = ax^2$. It opens upward if $p > 0$ and downward if $p < 0$ [see Figure 4, parts (a) and (b)]. The graph is symmetric with respect to the y-axis because (1) is unchanged when $x$ is replaced by $-x$.

If we interchange $x$ and $y$ in (1), we obtain

$$y^2 = 4px$$

which is an equation of the parabola with focus $(p, 0)$ and directrix $x = -p$. (Interchanging $x$ and $y$ amounts to reflecting about the diagonal line $y = x$.) The parabola opens to the right if $p > 0$ and to the left if $p < 0$ [see Figure 4, parts (c) and (d)]. In both cases the graph is symmetric with respect to the $x$-axis, which is the axis of the parabola.

**EXAMPLE 1** Find the focus and directrix of the parabola $y^2 + 10x = 0$ and sketch the graph.

**SOLUTION** If we write the equation as $y^2 = -10x$ and compare it with Equation 2, we see that $4p = -10$, so $p = -\frac{5}{2}$. Thus the focus is $(p, 0) = (-\frac{5}{2}, 0)$ and the directrix is $x = \frac{5}{2}$. The sketch is shown in Figure 5.

### ELLIPSES

An ellipse is the set of points in a plane the sum of whose distances from two fixed points $F_1$ and $F_2$ is a constant (see Figure 6). These two fixed points are called the foci (plural of focus). One of Kepler’s laws is that the orbits of the planets in the solar system are ellipses with the Sun at one focus.

In order to obtain the simplest equation for an ellipse, we place the foci on the $x$-axis at the points $(-c, 0)$ and $(c, 0)$ as in Figure 7 so that the origin is halfway between the foci. Let the sum of the distances from a point on the ellipse to the foci be $2a > 0$. Then $P(x, y)$ is a point on the ellipse when

$$|PF_1| + |PF_2| = 2a$$

that is,

$$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a$$

or

$$\sqrt{(x-c)^2 + y^2} = 2a - \sqrt{(x+c)^2 + y^2}$$

Squaring both sides, we have

$$x^2 - 2cx + c^2 + y^2 = 4a^2 - 4a\sqrt{(x+c)^2 + y^2} + x^2 + 2cx + c^2 + y^2$$

which simplifies to

$$a\sqrt{(x+c)^2 + y^2} = a^2 + cx$$

We square again:

$$a^2(x^2 + 2cx + c^2 + y^2) = a^4 + 2a^2cx + c^2x^2$$

which becomes

$$(a^2 - c^2)x^2 + 2ay^2 = a^2(a^2 - c^2)$$
From triangle $F_1F_2P$ in Figure 7 we see that $2c < 2a$, so $c < a$ and, therefore, $a^2 - c^2 > 0$. For convenience, let $b^2 = a^2 - c^2$. Then the equation of the ellipse becomes $b^2x^2 + a^2y^2 = a^2b^2$ or, if both sides are divided by $a^2b^2$,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Since $b^2 = a^2 - c^2 < a^2$, it follows that $b < a$. The $x$-intercepts are found by setting $y = 0$. Then $x^2/a^2 = 1$, or $x^2 = a^2$, so $x = \pm a$. The corresponding points $(a, 0)$ and $(-a, 0)$ are called the vertices of the ellipse and the line segment joining the vertices is called the major axis. To find the $y$-intercepts we set $x = 0$ and obtain $y^2 = b^2$, so $y = \pm b$. Equation 3 is unchanged if $x$ is replaced by $-x$ or $y$ is replaced by $-y$, so the ellipse is symmetric about both axes. Notice that if the foci coincide, then $c = 0$, so $a = b$ and the ellipse becomes a circle with radius $r = a = b$.

We summarize this discussion as follows (see also Figure 8).

4. The ellipse

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \quad a \geq b > 0$$

has foci $(\pm c, 0)$, where $c^2 = a^2 - b^2$, and vertices $(\pm a, 0)$.

If the foci of an ellipse are located on the $y$-axis at $(0, \pm c)$, then we can find its equation by interchanging $x$ and $y$ in (4). (See Figure 9.)

5. The ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad a \geq b > 0$$

has foci $(0, \pm c)$, where $c^2 = a^2 - b^2$, and vertices $(0, \pm a)$.

**Example 2** Sketch the graph of $9x^2 + 16y^2 = 144$ and locate the foci.

**Solution** Divide both sides of the equation by 144:

$$\frac{x^2}{16} + \frac{y^2}{9} = 1$$

The equation is now in the standard form for an ellipse, so we have $a^2 = 16$, $b^2 = 9$, $a = 4$, and $b = 3$. The $x$-intercepts are $\pm 4$ and the $y$-intercepts are $\pm 3$. Also, $c^2 = a^2 - b^2 = 7$, so $c = \sqrt{7}$ and the foci are $(\pm \sqrt{7}, 0)$. The graph is sketched in Figure 10.

**Example 3** Find an equation of the ellipse with foci $(0, \pm 2)$ and vertices $(0, \pm 3)$.

**Solution** Using the notation of (5), we have $c = 2$ and $a = 3$. Then we obtain $b^2 = a^2 - c^2 = 9 - 4 = 5$, so an equation of the ellipse is

$$\frac{x^2}{5} + \frac{y^2}{9} = 1$$

Another way of writing the equation is $9x^2 + 5y^2 = 45$.

Like parabolas, ellipses have an interesting reflection property that has practical consequences. If a source of light or sound is placed at one focus of a surface with elliptical cross-sections, then all the light or sound is reflected off the surface to the other focus (see
A hyperbola is the set of all points in a plane the difference of whose distances from two fixed points \(F_1\) and \(F_2\) (the foci) is a constant. This definition is illustrated in Figure 11.

Hyperbolas occur frequently as graphs of equations in chemistry, physics, biology, and economics (Boyle’s Law, Ohm’s Law, supply and demand curves). A particularly significant application of hyperbolas is found in the navigation systems developed in World Wars I and II (see Exercise 51).

Notice that the definition of a hyperbola is similar to that of an ellipse; the only change is that the sum of distances has become a difference of distances. In fact, the derivation of the equation of a hyperbola is also similar to the one given earlier for an ellipse. It is left as Exercise 52 to show that when the foci are on the \(x\)-axis at \((\pm c, 0)\) and the difference of distances is \(|PF_1| - |PF_2| = \pm 2a\), then the equation of the hyperbola is

\[
\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1
\]

where \(c^2 = a^2 + b^2\). Notice that the \(x\)-intercepts are again \(\pm a\) and the points \((a, 0)\) and \((-a, 0)\) are the vertices of the hyperbola. But if we put \(x = 0\) in Equation 6 we get \(y^2 = -b^2\), which is impossible, so there is no \(y\)-intercept. The hyperbola is symmetric with respect to both axes.

To analyze the hyperbola further, we look at Equation 6 and obtain

\[
\frac{x^2}{a^2} = 1 + \frac{y^2}{b^2} \Rightarrow 1
\]

This shows that \(x^2 \geq a^2\), so \(|x| = \sqrt{x^2} \geq a\). Therefore, we have \(x \geq a\) or \(x \leq -a\). This means that the hyperbola consists of two parts, called its branches.

When we draw a hyperbola it is useful to first draw its asymptotes, which are the dashed lines \(y = (b/a)x\) and \(y = -(b/a)x\) shown in Figure 12. Both branches of the hyperbola approach the asymptotes; that is, they come arbitrarily close to the asymptotes.

If the foci of a hyperbola are on the \(y\)-axis, then by reversing the roles of \(x\) and \(y\) we obtain the following information, which is illustrated in Figure 13.

The hyperbola

\[
\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1
\]

has foci \((0, \pm c)\), where \(c^2 = a^2 + b^2\), vertices \((0, \pm a)\), and asymptotes \(y = \pm (a/b)x\).
EXAMPLE 4 Find the foci and asymptotes of the hyperbola \(9x^2 - 16y^2 = 144\) and sketch its graph.

**SOLUTION** If we divide both sides of the equation by 144, it becomes

\[
\frac{x^2}{16} - \frac{y^2}{9} = 1
\]

which is of the form given in (7) with \(a = 4\) and \(b = 3\). Since \(c^2 = 16 + 9 = 25\), the foci are \((\pm 5, 0)\). The asymptotes are the lines \(y = \frac{3}{4}x\) and \(y = -\frac{3}{4}x\). The graph is shown in Figure 14.

EXAMPLE 5 Find the foci and equation of the hyperbola with vertices \((0, \pm 1)\) and asymptote \(y = 2x\).

**SOLUTION** From (8) and the given information, we see that \(a = 1\) and \(a/b = 2\). Thus, \(b = a/2 = \frac{1}{2}\) and \(c^2 = a^2 + b^2 = \frac{5}{4}\). The foci are \((0, \pm \sqrt{5}/2)\) and the equation of the hyperbola is

\[y^2 - 4x^2 = 1\]

**SHIFTED CONICS**

We shift conics by taking the standard equations (1), (2), (4), (5), (7), and (8) and replacing \(x\) and \(y\) by \(x - h\) and \(y - k\).

EXAMPLE 6 Find an equation of the ellipse with foci \((2, -2), (4, -2)\) and vertices \((1, -2), (5, -2)\).

**SOLUTION** The major axis is the line segment that joins the vertices \((1, -2), (5, -2)\) and has length 4, so \(a = 2\). The distance between the foci is 2, so \(c = 1\). Thus, \(b^2 = a^2 - c^2 = 3\). Since the center of the ellipse is \((3, -2)\), we replace \(x\) and \(y\) in (4) by \(x - 3\) and \(y + 2\) to obtain

\[
\frac{(x - 3)^2}{4} + \frac{(y + 2)^2}{3} = 1
\]

as the equation of the ellipse.

EXAMPLE 7 Sketch the conic

\[9x^2 - 4y^2 - 72x + 8y + 176 = 0\]

and find its foci.

**SOLUTION** We complete the squares as follows:

\[
4(y^2 - 2y) - 9(x^2 - 8x) = 176
\]

\[4(y^2 - 2y + 1) - 9(x^2 - 8x + 16) = 176 + 4 - 144
\]

\[4(y - 1)^2 - 9(x - 4)^2 = 36
\]

\[\frac{(y - 1)^2}{9} - \frac{(x - 4)^2}{4} = 1
\]

This is in the form (8) except that \(x\) and \(y\) are replaced by \(x - 4\) and \(y - 1\). Thus, \(a^2 = 9, b^2 = 4, \) and \(c^2 = 13\). The hyperbola is shifted four units to the right and one unit upward. The foci are \((4, 1 + \sqrt{13})\) and \((4, 1 - \sqrt{13})\) and the vertices are \((4, 4)\) and \((4, -2)\). The asymptotes are \(y - 1 = \pm \frac{3}{2}(x - 4)\). The hyperbola is sketched in Figure 15.
EXERCISES

1–8  Find the vertex, focus, and directrix of the parabola and sketch its graph.

1. \(x = 2y^2\)
2. \(4y + x^2 = 0\)
3. \(4x^2 = -y\)
4. \(y^2 = 12x\)
5. \((x + 2)^2 = 8(y - 3)\)
6. \(x - 1 = (y + 5)^2\)
7. \(y^2 + 2y + 12x + 25 = 0\)
8. \(y + 12x - 2x^2 = 16\)

9–10  Find an equation of the parabola. Then find the focus and directrix.

9. 

10. 

9. \(x^2 = 4y\)
10. \(y^2 = 4x\)

11–16  Find the vertices and foci of the ellipse and sketch its graph.

11. \(\frac{x^2}{9} + \frac{y^2}{5} = 1\)
12. \(\frac{x^2}{64} + \frac{y^2}{100} = 1\)
13. \(4x^2 + y^2 = 16\)
14. \(4x^2 + 25y^2 = 25\)
15. \(9x^2 - 18x + 4y^2 = 27\)
16. \(x^2 + 2y^2 - 6x + 4y + 7 = 0\)

17–18  Find an equation of the ellipse. Then find its foci.

17. 

18. 

17. \(\frac{x^2}{144} - \frac{y^2}{25} = 1\)
18. \(\frac{y^2}{16} - \frac{x^2}{36} = 1\)

19–20  Find the vertices, foci, and asymptotes of the hyperbola and sketch its graph.

19. \(\frac{x^2}{144} - \frac{y^2}{25} = 1\)
20. \(\frac{y^2}{16} - \frac{x^2}{36} = 1\)
21. \(y^2 - x^2 = 4\)
22. \(9x^2 - 4y^2 = 36\)

23. \(2y^2 - 3x^2 - 4y + 12x + 8 = 0\)
24. \(16x^2 - 9y^2 + 64x - 90y = 305\)

25–30  Identify the type of conic section whose equation is given and find the vertices and foci.

25. \(x^2 = y + 1\)
26. \(x^2 = y^2 + 1\)
27. \(x^2 = 4y - 2y^2\)
28. \(y^2 - 8y = 6x - 16\)
29. \(y^2 + 2y = 4x^2 + 3\)
30. \(4x^2 + 4x + y^2 = 0\)

31–48  Find an equation for the conic that satisfies the given conditions.

31. Parabola, vertex (0, 0), focus (0, -2)
32. Parabola, vertex (1, 0), directrix \(x = -5\)
33. Parabola, focus (-4, 0), directrix \(x = 2\)
34. Parabola, focus (3, 6), vertex (3, 2)
35. Parabola, vertex (0, 0), axis the x-axis, passing through (1, -4)
36. Parabola, vertical axis, passing through (-2, 3), (0, 3), and (1, 9)
37. Ellipse, foci (-2, 0), vertices (-5, 0)
38. Ellipse, foci (0, ±5), vertices (0, ±13)
39. Ellipse, foci (0, 2), (0, 6), vertices (0, 0), (0, 8)
40. Ellipse, foci (0, -1), (8, -1), vertex (9, -1)
41. Ellipse, center (2, 2), focus (0, 2), vertex (5, 2)
42. Ellipse, foci (±2, 0), passing through (2, 1)
43. Hyperbola, foci (0, ±3), vertices (0, ±1)
44. Hyperbola, foci (±6, 0), vertices (±4, 0)
45. Hyperbola, foci (1, 3) and (7, 3), vertices (2, 3) and (6, 3)
46. Hyperbola, foci (2, -2) and (2, 8), vertices (2, 0) and (2, 6)
47. Hyperbola, vertices (±3, 0), asymptotes \(y = ±2x\)
48. Hyperbola, foci (2, 2) and (6, 2), asymptotes \(y = x - 2\) and \(y = 6 - x\)

49. The point in a lunar orbit nearest the surface of the moon is called perilune and the point farthest from the surface is called apolune. The Apollo 11 spacecraft was placed in an elliptical lunar orbit with perilune altitude 110 km and apolune altitude 314 km (above the moon). Find an equation of this ellipse if the radius of the moon is 1728 km and the center of the moon is at one focus.
50. A cross-section of a parabolic reflector is shown in the figure. The bulb is located at the focus and the opening at the focus is 10 cm.
(a) Find an equation of the parabola.
(b) Find the diameter of the opening \( |CD| \), 11 cm from the vertex.

51. In the LORAN (LOng RAnge Navigation) radio navigation system, two radio stations located at \( A \) and \( B \) transmit simultaneous signals to a ship or an aircraft located at \( P \). The onboard computer converts the time difference in receiving these signals into a distance difference \( |PA| - |PB| \), and this, according to the definition of a hyperbola, locates the ship or aircraft on one branch of a hyperbola (see the figure). Suppose that station \( B \) is located 400 mi due east of station \( A \) on a coastline. A ship received the signal from \( B \) 1200 microseconds (\( \mu s \)) before it received the signal from \( A \).
(a) Assuming that radio signals travel at a speed of 980 ft/\( \mu s \), find an equation of the hyperbola on which the ship lies.
(b) If the ship is due north of \( B \), how far off the coastline is the ship?

52. Use the definition of a hyperbola to derive Equation 6 for a hyperbola with foci \(( \pm c, 0)\) and vertices \(( \pm a, 0)\).

53. Show that the function defined by the upper branch of the hyperbola \( \frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \) is concave upward.

54. Find an equation for the ellipse with foci \((1, 1)\) and \((-1, -1)\) and major axis of length 4.

55. Determine the type of curve represented by the equation
\[
\frac{x^2}{k} + \frac{y^2}{k - 16} = 1
\]
in each of the following cases: (a) \( k > 16 \), (b) \( 0 < k < 16 \), and (c) \( k < 0 \).
(d) Show that all the curves in parts (a) and (b) have the same foci, no matter what the value of \( k \) is.

56. (a) Show that the equation of the tangent line to the parabola \( y^2 = 4px \) at the point \((x_0, y_0)\) can be written as \( y_0y = 2px + x_0 \).
(b) What is the x-intercept of this tangent line? Use this fact to draw the tangent line.

57. Use Simpson’s Rule with \( n = 10 \) to estimate the length of the ellipse \( x^2 + 4y^2 = 4 \).

58. The planet Pluto travels in an elliptical orbit around the Sun (at one focus). The length of the major axis is \( 1.18 \times 10^{10} \) km and the length of the minor axis is \( 1.14 \times 10^{10} \) km. Use Simpson’s Rule with \( n = 10 \) to estimate the distance traveled by the planet during one complete orbit around the Sun.

59. Let \((x_1, y_1)\) be a point on the ellipse \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \) with foci \( F_1 \) and \( F_2 \) and let \( \alpha \) and \( \beta \) be the angles between the lines \( PF_1, PF_2 \) and the ellipse as in the figure. Prove that \( \alpha = \beta \). This explains how whispering galleries and lithotripsy work. Sound coming from one focus is reflected and passes through the other focus. [Hint: Use the formula \( \tan \alpha = \frac{m_2 - m_1}{1 + m_2m_1} \) to show that \( \tan \alpha = \tan \beta \). See Challenge Problem 2.13.]

60. Let \((x_1, y_1)\) be a point on the hyperbola \( \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \) with foci \( F_1 \) and \( F_2 \) and let \( \alpha \) and \( \beta \) be the angles between the lines \( PF_1, PF_2 \) and the hyperbola as shown in the figure. Prove that \( \alpha = \beta \). (This is the reflection property of the hyperbola. It shows that light aimed at a focus \( F_2 \) of a hyperbolic mirror is reflected toward the other focus \( F_1 \).)