Combinatorics and representation theory of diagram algebras.

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Slides available at https://zdaugherty.ccnysites.cuny.edu/research/
Combinatorial representation theory

Given an algebra $A$, what are the $A$-modules/representations? (Actions $A \times V$ and homomorphisms $\varphi: A \to \text{End}_\mathbb{k}V$)

What are the simple/indecomposable $A$-modules/reps? What are their dimensions?

What is the action of the center of $A$?

How can I combine modules to make new ones, and what are they in terms of the simple modules?

In combinatorial representation theory, we use combinatorial objects to index (construct a bijection to) modules and representations, and to encode information about them.
Combinatorial representation theory

Representation theory: Given an algebra $A$...

- What are the $A$-modules/representations?
  \[(\text{Actions } A \odot V \text{ and homomorphisms } \varphi : A \to \text{End}(V))\]
- What are the simple/indecomposable $A$-modules/reps?
- What are their dimensions?
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Combinatorial representation theory

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In combinatorial representation theory, we use combinatorial objects to index (construct a bijection to) modules and representations, and to encode information about them.
Motivating example: Schur-Weyl Duality

The **symmetric group** $S_k$ (permutations) as diagrams:
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(With multiplication given by concatenation)
Motivating example: Schur-Weyl Duality

The **symmetric group** $S_k$ (permutations) as diagrams:

![Diagram of the symmetric group $S_k$]

(with multiplication given by concatenation)
Motivating example: Schur-Weyl Duality

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Motivating example: Schur-Weyl Duality

\[ \text{GL}_n(\mathbb{C}) \text{ acts on } \bigotimes \mathbb{C}^n = (\mathbb{C}^n)^\otimes_k \text{ diagonally.} \]

\[ g \cdot (v_1 \otimes v_2 \otimes \cdots \otimes v_k) = gv_1 \otimes gv_2 \otimes \cdots \otimes gv_k. \]
Motivating example: Schur-Weyl Duality

$GL_n(\mathbb{C})$ acts on $\mathbb{C}^n \otimes \mathbb{C}^n \otimes \cdots \otimes \mathbb{C}^n = (\mathbb{C}^n)^\otimes k$ diagonally.

$$g \cdot (v_1 \otimes v_2 \otimes \cdots \otimes v_k) = g v_1 \otimes g v_2 \otimes \cdots \otimes g v_k.$$

$S_k$ also acts on $(\mathbb{C}^n)^\otimes k$ by place permutation.
Motivating example: Schur-Weyl Duality

\( \text{GL}_n(\mathbb{C}) \) acts on \( \mathbb{C}^n \otimes \mathbb{C}^n \otimes \cdots \otimes \mathbb{C}^n = (\mathbb{C}^n)^{\otimes k} \) diagonally.

\[
g \cdot (v_1 \otimes v_2 \otimes \cdots \otimes v_k) = gv_1 \otimes gv_2 \otimes \cdots \otimes gv_k.
\]

\( S_k \) also acts on \( (\mathbb{C}^n)^{\otimes k} \) by place permutation.

These actions commute!
Motivating example: Schur-Weyl Duality

Schur (1901): $S_k$ and $\text{GL}_n$ have commuting actions on $(\mathbb{C}^n)^\otimes k$.

Even better,

$$\text{End}_{\text{GL}_n} \left( (\mathbb{C}^n)^\otimes k \right) = \pi(\mathbb{C}S_k) \quad \text{and} \quad \text{End}_{S_k} \left( (\mathbb{C}^n)^\otimes k \right) = \rho(\mathbb{C}\text{GL}_n).$$

(all linear maps that commute with $\text{GL}_n$) (img of $S_k$ action)

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(all linear maps that commute with $GL_n$)\hspace{1cm} (img of $S_k$ action)

Powerful consequence:

The double-centralizer relationship produces

$$\left(\mathbb{C}^n\right)^\otimes k \cong \bigoplus_{\lambda \vdash k} G^\lambda \otimes S^\lambda \quad \text{as a } GL_n - S_k \text{ bimodule},$$

where $G^\lambda$ are distinct irreducible $GL_n$-modules

$S^\lambda$ are distinct irreducible $S_k$-modules
Motivating example: Schur-Weyl Duality

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Even better,

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(All linear maps that commute with $\text{GL}_n$) (img of $S_k$ action) (img of $\text{GL}_n$ action)

Powerful consequence:

The double-centralizer relationship produces

$$(\mathbb{C}^n)^\otimes k \cong \bigoplus_{\lambda \vdash k} G^\lambda \otimes S^\lambda$$

as a $\text{GL}_n$-$S_k$ bimodule,

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For example,

$$\mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n = \left( G \otimes S \right) \oplus \left( G \otimes S \right) \oplus \left( G \otimes S \right)$$
Representation theory of $V^\otimes k$

$V = \mathbb{C} = L(\square)$
Representation theory of $V^\otimes k$

$V = \mathbb{C} = L(\square)$

$L(\square)$
Representation theory of $V^\otimes k$

$V = \mathbb{C} = L(\square)$, \quad $L(\square) \otimes L(\square)$

\[ \begin{array}{c}
\emptyset \\
\downarrow \\
\square \\
\downarrow \\
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\square \\
\square \\
\end{array} \]
Representation theory of $V^\otimes k$

$V = \mathbb{C} = L(\square)$, \quad $L(\square) \otimes L(\square) \otimes L(\square)$
Representation theory of $V^\otimes k$

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Representation theory of $V^\otimes k$

$V = \mathbb{C} = L(\square)$, \quad $L(\square) \otimes L(\square) \otimes L(\square) \otimes L(\square) \otimes L(\square) \cdots$

\[ \emptyset \]

\[ \vdots \]

\[ \vdots \]

\[ \vdots \]
Representation theory of $V^\otimes k$

$$V = \mathbb{C} = L(\square), \quad L(\square) \otimes L(\square) \otimes L(\square) \otimes L(\square) \otimes L(\square) \cdots$$
Representation theory of $V^\otimes k$

$$V = \mathbb{C} = L(\square), \quad L(\square) \otimes L(\square) \otimes L(\square) \otimes L(\square) \otimes L(\square) \cdots$$
More centralizer algebras

Brauer (1937)
Orthogonal and symplectic groups
(and Lie algebras) acting on
\((\mathbb{C}^n)^\otimes k\) diagonally centralize
the **Brauer algebra**:

\[
\delta_{b,c} \sum_{i=1}^{n} v_i \otimes v_i \otimes v_a \otimes v_d \otimes v_d \\
\]

with \(\bigcirc = n\)

Diagrams encode maps \(V^\otimes k \to V^\otimes k\) that commute with the
action of some classical algebra.
More centralizer algebras

Representation theory of $V^\otimes k$, orthogonal and symplectic:

$V = \mathbb{C} = L(\square)$
More centralizer algebras

Representation theory of $V^\otimes k$, orthogonal and symplectic:

$V = \mathbb{C} = L(\square)$, $L(\square)$

$\varnothing$

$\square$
More centralizer algebras

Representation theory of $V^\otimes k$, orthogonal and symplectic:

$V = \mathbb{C} = L(\Box), \quad L(\Box) \otimes L(\Box)$

\[ \emptyset \]

\[ \begin{array}{c}
\emptyset \\
\Box \\
\emptyset \quad \Box \\
\end{array} \]
More centralizer algebras

Representation theory of $V^\otimes k$, orthogonal and symplectic:

$V = \mathbb{C} = L(\square), \quad L(\square) \otimes L(\square) \otimes L(\square)$
More centralizer algebras

Representation theory of $V^\otimes k$, orthogonal and symplectic:

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More centralizer algebras

Brauer (1937)
Orthogonal and symplectic groups (and Lie algebras) acting on \((\mathbb{C}^n)^\otimes k\) diagonally centralize the Brauer algebra:

\[
\delta_{b,c} \sum_{i=1}^{n} v_i \otimes v_i \otimes v_a \otimes v_d \otimes v_d
\]
with \(\bigcirc = n\)

Temperley-Lieb (1971)
\(\text{GL}_2\) and \(\text{SL}_2\) (and \(\mathfrak{gl}_2\) and \(\mathfrak{sl}_2\)) acting on \((\mathbb{C}^2)^\otimes k\) diagonally centralize the Temperley-Lieb algebra:

\[
\delta_{c,d} \sum_{i=1}^{2} v_a \otimes v_i \otimes v_i \otimes v_b \otimes v_e
\]
with \(\bigcirc = 2\)

Diagrams encode maps \(V^\otimes k \rightarrow V^\otimes k\) that commute with the action of some classical algebra.
More diagram algebras: braids

The **braid group**:

(with multiplication given by concatenation)
More diagram algebras: braids

The **braid group**:

(With multiplication given by concatenation)
More diagram algebras: braids

The affine (one-pole) braid group:

(with multiplication given by concatenation)
Quantum groups and braids

Fix $q \in \mathbb{C}$, and let $\mathcal{U} = \mathcal{U}_q g$ be the Drinfeld-Jimbo quantum group associated to Lie algebra $g$. 
Quantum groups and braids

Fix $q \in \mathbb{C}$, and let $\mathcal{U} = \mathcal{U}_q \mathfrak{g}$ be the Drinfeld-Jimbo quantum group associated to Lie algebra $\mathfrak{g}$.

$\mathcal{U} \otimes \mathcal{U}$ has an invertible element $\mathcal{R} = \sum_{\mathcal{R}} R_1 \otimes R_2$ that yields a map

$$\tilde{\mathcal{R}}_{VW} : V \otimes W \longrightarrow W \otimes V$$

that

1. satisfies braid relations, and
2. commutes with the action on $V \otimes W$

for any $\mathcal{U}$-module $V$. 
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The braid group shares a commuting action with $\mathcal{U}$ on $V^\otimes k$:
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that (1) satisfies braid relations, and

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for any $\mathcal{U}$-module $V$.

The one-pole/affine braid group shares a commuting action with $\mathcal{U}$ on $M \otimes V^\otimes k$:

Around the pole:

$$= \tilde{R}_{MV} \tilde{R}_{VM}$$
Quantum groups and braids

Fix \( q \in \mathbb{C} \), and let \( \mathcal{U} = \mathcal{U}_q \mathfrak{g} \) be the Drinfeld-Jimbo quantum group associated to Lie algebra \( \mathfrak{g} \).

\( \mathcal{U} \otimes \mathcal{U} \) has an invertible element \( \mathcal{R} = \sum_{\mathcal{R}} R_1 \otimes R_2 \) that yields a map

\[
\tilde{\mathcal{R}}_{VW} : V \otimes W \longrightarrow W \otimes V
\]

that
1. satisfies braid relations, and
2. commutes with the action on \( V \otimes W \) for any \( \mathcal{U} \)-module \( V \).

The two-pole braid group shares a commuting action with \( \mathcal{U} \) on

\[
M \otimes V^\otimes k \otimes N:
\]

Around the pole:

\[
= \tilde{R}_{MV} \tilde{R}_{VM}
\]
Universal

Type B, C, D
(orthog. & sympl.)

Type A
(gen. & sp. linear)

Small Type A
(GL$_2$ & SL$_2$)

Lie grp/alg

Brauer algebra

Sym. group

Temperley-Lieb

Braid group

BMW algebra

Hecke algebra

\[ \mathcal{H} = a \mathcal{H} + \mathbb{I} \]

Affine braids

Affine BMW

Affine Hecke of type A (+twists)

One-boundary TL

Affine Hecke of type C (+twists)

Two-boundary TL
<table>
<thead>
<tr>
<th>Qu. grps:</th>
<th>Lie algs:</th>
</tr>
</thead>
<tbody>
<tr>
<td>Orthogonal and symplectic (types B, C, D)</td>
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</tr>
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</table>

- **$V \otimes^k$**
  - **BMW algebra**
  - **Brauer algebra**

- **$M \otimes V \otimes^k$**
  - **Affine BMW**
  - **Deg. aff. BMW**

- **$M \otimes V \otimes^k \otimes N$**
  - **2-bdry BMW**
  - **Deg. 2-bdry BMW**

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H" aring-Oldenburg (98) and Orellana-Ram (04): Introduced the affine BMW algebras. [OR04] gave the action on $M \otimes V \otimes^k$ commuting with the action of the quantum groups of types B, C, D.

D.-Ram-Virk: Used these centralizer relationships to study these two algebras simultaneously. Results include computing the centers, handling the parameters associated to the algebras, computing powerful intertwiner operators, etc.

D.-Gonz´ alez-Schneider-Sutton: Constructing 2-boundary analogues (in progress).

Balagovic et al.: Signed versions and representations of periplectic Lie superalgebras.
Orthogonal and symplectic (types B, C, D)

Qu. grps: BMW algebra

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Othogonal and symplectic (types B, C, D)

Quantum groups:
- BMW algebra
- Affine BMW algebra
- 2-boundary BMW algebra

Lie algebras:
- Brauer algebra
- Degenerate affine BMW algebra
- Degenerate 2-boundary BMW algebra


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Example: “Admissibility conditions”

**Affine BMW algebra**

Closed loops:

**Degenerate affine BMW algebra**

Closed loops:

The associated parameters of the algebra, e.g. $z_0$, $z_1$, $z_2$, … aren’t entirely free.

Important insight: As operators on tensor space $M \otimes V \otimes V \otimes Z \otimes U \otimes g \otimes C \otimes C$ and $\ell \otimes Z \otimes p \otimes U \otimes q \otimes g \otimes C \otimes C$. 

“Higher Casimir invariants”
Example: “Admissibility conditions”

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\[ z_0, \quad z_1, \quad z_2, \quad \ldots \]

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“Higher Casimir invariants”

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Qu. grps: 

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Universal Type B, C, D
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Lie grp/alg

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V = ⧲

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= aH + ℐ

Affine braids

Affine BMW

Affine Hecke
of type A
(+ twists)

One-boundary TL

Affine Hecke
of type C
(+ twists)

Two-boundary TL

Two-pole braids

Two-pole BMW

Affine Hecke
of type C
(+ twists)

Two-boundary TL
Two boundary algebras (type A)  
Nienhuis, de Gier, Batchelor (2004): Studying the six-vertex model with additional integrable boundary terms, introduced the two-boundary Temperley-Lieb algebra $TL_k$:  

Even # dots non-crossing diagrams  

de Gier, Nichols (2008): Explored representation theory of $TL_k$ using diagrams and established a connection to the affine Hecke algebras of type A and C.

D. (2010): The centralizer of $\mathfrak{gl}_n$ acting on tensor space $M \otimes V \otimes N$ displays type C combinatorics for good choices of $M$, $N$, and $V$. 
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Two-pole braids
Two-pole BMW
Affine Hecke of type C (+twists)
Two-boundary TL
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- Non-crossing diagrams
  - Even number of dots
  - $k$ dots

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**Two boundary algebras (type A)**

Nienhuis, de Gier, Batchelor (2004): Studying the six-vertex model with additional integrable boundary terms, introduced the two-boundary Temperley-Lieb algebra $TL_k$:

![Diagram of non-crossing diagrams](image)

Non-crossing diagrams with even number of dots and $k$ dots.

de Gier, Nichols (2008): Explored representation theory of $TL_k$ using diagrams and established a connection to the affine Hecke algebras of type A and C.

D. (2010): The centralizer of $\mathfrak{gl}_n$ acting on tensor space $M \otimes V^k \otimes N$ displays type C combinatorics for good choices of $M$, $N$, and $V$. 
The two-boundary (two-pole) braid group $B_k$ is generated by

$$T_k = \begin{array}{c}
\begin{array}{c}
\text{Diagram}
\end{array}
\end{array}, \quad T_0 = \begin{array}{c}
\begin{array}{c}
\text{Diagram}
\end{array}
\end{array} \quad \text{and} \quad T_i = \begin{array}{c}
\begin{array}{c}
\text{Diagram}
\end{array}
\end{array} \quad \text{for} \ 1 \leq i \leq k - 1,$$

subject to relations

$$T_0 T_1 T_2 \cdots T_k = T_k T_1 T_0 = 1,$$

and, similarly,

$$T_k T_1 T_k \cdots T_{k-1} = T_1 T_k T_1 = 1.$$
The two-boundary (two-pole) braid group $\mathcal{B}_k$ is generated by

$$T_k = \begin{array}{c}
\includegraphics{braid Tk}
\end{array}, \quad T_0 = \begin{array}{c}
\includegraphics{braid T0}
\end{array} \quad \text{and} \quad T_i = \begin{array}{c}
\includegraphics{braid Ti}
\end{array} \quad \text{for } 1 \leq i \leq k - 1,$$

subject to relations

$$T_i T_{i+1} T_i = \begin{array}{c}
\includegraphics{braid TIT}
\end{array} = \begin{array}{c}
\includegraphics{braid TTT}
\end{array} = T_{i+1} T_i T_{i+1},$$
The two-boundary (two-pole) braid group $\mathcal{B}_k$ is generated by

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\[ T_i T_{i+1} T_i = \begin{array}{c} \text{Diagram} \end{array} = \begin{array}{c} \text{Diagram} \end{array} = T_{i+1} T_i T_{i+1}, \]

\[ T_1 T_0 T_1 T_0 = \begin{array}{c} \text{Diagram} \end{array} = \begin{array}{c} \text{Diagram} \end{array} = T_0 T_1 T_0 T_1, \]
The two-boundary (two-pole) braid group $\mathcal{B}_k$ is generated by

$$T_k = \includegraphics{boundary_braid_k}, \quad T_0 = \includegraphics{boundary_braid_0} \quad \text{and} \quad T_i = \includegraphics{boundary_braid_i} \quad \text{for} \ 1 \leq i \leq k - 1,$$

subject to relations

$$T_i T_{i+1} T_i = \includegraphics{relation_i}, \quad T_1 T_0 T_1 T_0 = \includegraphics{relation_0}, \quad \text{and, similarly,} \quad T_{k-1} T_k T_{k-1} T_k = \includegraphics{relation_k}.$$
The two-boundary (two-pole) braid group $\mathcal{B}_k$ is generated by

\[ T_k = \begin{array}{c}
\includegraphics[width=0.1\textwidth]{tk.pdf}
\end{array} \quad T_0 = \begin{array}{c}
\includegraphics[width=0.1\textwidth]{t0.pdf}
\end{array} \quad \text{and} \quad T_i = \begin{array}{c}
\includegraphics[width=0.1\textwidth]{ti.pdf}
\end{array} \quad \text{for } 1 \leq i \leq k - 1,
\]

subject to relations

\[ T_0 T_1 T_2 = T_{k-2} T_{k-1} T_k \]

i.e.

\[ T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \]
\[ T_1 T_0 T_1 T_0 = T_0 T_1 T_0 T_1 \]

and, similarly, \( T_{k-1} T_k T_{k-1} T_k = T_k T_{k-1} T_k T_{k-1} \).
(1) The two-boundary (two-pole) braid group $B_k$ is generated by

$$T_k = \begin{array}{c}
\includegraphics{tk}\end{array}, \quad T_0 = \begin{array}{c}
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\includegraphics{ti}\end{array} \quad \text{for } 1 \leq i \leq k - 1,$$

subject to relations

$$T_0 T_1 T_2 \ldots T_{k-2} T_{k-1} T_k.$$
(1) The two-boundary (two-pole) braid group $B_k$ is generated by

$$T_k = \begin{array}{c}
\text{\includegraphics[width=1cm]{Diagram1.png}}
\end{array}, \quad T_0 = \begin{array}{c}
\text{\includegraphics[width=1cm]{Diagram2.png}}
\end{array} \quad \text{and} \quad T_i = \begin{array}{c}
\text{\includegraphics[width=1cm]{Diagram3.png}}
\end{array} \quad \text{for} \ 1 \leq i \leq k - 1,$$

subject to relations

$$\begin{array}{c}
\text{\includegraphics[width=10cm]{Relations.png}}
\end{array}.$$ 

(2) Fix constants $t_0, t_k, t \in \mathbb{C}$. The affine type C Hecke algebra $\mathcal{H}_k$ is the quotient of $\mathbb{C} B_k$ by the relations

$$(T_0 - t_0^{1/2})(T_0 + t_0^{-1/2}) = 0, \quad (T_k - t_k^{1/2})(T_k + t_k^{-1/2}) = 0$$

and

$$(T_i - t^{1/2})(T_i + t^{-1/2}) = 0 \quad \text{for} \ 1 \leq t \leq k - 1.$$
(1) The two-boundary (two-pole) braid group $B_k$ is generated by

$$T_k = \begin{array}{c}
\begin{array}{c}
\bullet \\
\end{array}
\end{array}, \quad T_0 = \begin{array}{c}
\begin{array}{c}
\bullet \\
\end{array}
\end{array} \quad \text{and} \quad T_i = \begin{array}{c}
\begin{array}{c}
\bullet \\
\end{array}
\end{array} \quad \text{for } 1 \leq i \leq k - 1,$$

subject to relations

$$\begin{array}{c}
\begin{array}{c}
\bullet \\
\end{array}
\end{array} \begin{array}{c}
\begin{array}{c}
\bullet \\
\end{array}
\end{array} \begin{array}{c}
\begin{array}{c}
\bullet \\
\end{array}
\end{array} \begin{array}{c}
\begin{array}{c}
\bullet \\
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\end{array} \begin{array}{c}
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\bullet \\
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\end{array} \begin{array}{c}
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\end{array} \begin{array}{c}
\begin{array}{c}
\bullet \\
\end{array}
\end{array} \begin{array}{c}
\begin{array}{c}
\bullet \\
\end{array}
\end{array} \begin{array}{c}
\begin{array}{c}
\bullet \\
\end{array}
\end{array}.$$

(2) Fix constants $t_0, t_k, t = t_1 = t_2 = \cdots = t_{k-1} \in \mathbb{C}$. The affine type C Hecke algebra $\mathcal{H}_k$ is the quotient of $\mathbb{C}B_k$ by the relations $(T_i - t_i^{1/2})(T_i + t_i^{-1/2}) = 0$. 
(1) The two-boundary (two-pole) braid group $B_k$ is generated by

$T_k = \begin{array}{c}
\text{\includegraphics[scale=0.5]{braid1.png}}
\end{array}$, \hspace{1em} $T_0 = \begin{array}{c}
\text{\includegraphics[scale=0.5]{braid2.png}}
\end{array}$ \text{ and } \begin{array}{c}
T_i = \begin{array}{c}
\text{\includegraphics[scale=0.5]{braid3.png}}
\end{array} \text{ for } 1 \leq i \leq k - 1,
\end{array}$

subject to relations

$\begin{array}{c}
\text{\includegraphics[scale=0.5]{relations1.png}}
\end{array}$.

(2) Fix constants $t_0, t_k, t = t_1 = t_2 = \cdots = t_{k-1} \in \mathbb{C}$. The affine type C Hecke algebra $\mathcal{H}_k$ is the quotient of $\mathbb{C}B_k$ by the relations $(T_i - t_i^{1/2})(T_i + t_i^{-1/2}) = 0$.

(3) Set

$\begin{array}{c}
\text{\includegraphics[scale=0.5]{hecke_relations1.png}} = t_i^{1/2} \begin{array}{c}
\text{\includegraphics[scale=0.5]{hecke_relations2.png}} - \begin{array}{c}
\text{\includegraphics[scale=0.5]{hecke_relations3.png}}
\end{array}
\end{array}
\end{array}$ \hspace{1em} (e_0 = t_0^{1/2} - T_0)

$\begin{array}{c}
\text{\includegraphics[scale=0.5]{hecke_relations4.png}} = t_i^{1/2} \begin{array}{c}
\text{\includegraphics[scale=0.5]{hecke_relations5.png}} - \begin{array}{c}
\text{\includegraphics[scale=0.5]{hecke_relations6.png}}
\end{array}
\end{array}
\end{array}$ \hspace{1em} (e_k = t_k^{1/2} - T_k)

$\begin{array}{c}
\text{\includegraphics[scale=0.5]{hecke_relations7.png}} = t_i^{1/2} \begin{array}{c}
\text{\includegraphics[scale=0.5]{hecke_relations8.png}} - \begin{array}{c}
\text{\includegraphics[scale=0.5]{hecke_relations9.png}}
\end{array}
\end{array}
\end{array}$ \hspace{1em} (e_i = t_i^{1/2} - T_i)

so that $e_j^2 = z_j e_j$ (for good $z_j$).
(1) The two-boundary (two-pole) braid group $B_k$ is generated by

$$T_k = \begin{array}{c}
\includegraphics[width=1cm]{tikz/tikz_braid.png}
\end{array}, \quad T_0 = \begin{array}{c}
\includegraphics[width=1cm]{tikz/tikz_braid.png}
\end{array} \quad \text{and} \quad T_i = \begin{array}{c}
\includegraphics[width=1cm]{tikz/tikz_braid.png}
\end{array} \quad \text{for } 1 \leq i \leq k - 1,$$

subject to relations

$$\begin{array}{cccccc}
T_0 & T_1 & T_2 & T_{k-2} & T_{k-1} & T_k
\end{array}.$$

(2) Fix constants $t_0, t_k, t = t_1 = t_2 = \cdots = t_{k-1} \in \mathbb{C}$. The affine type C Hecke algebra $H_k$ is the quotient of $\mathbb{C}B_k$ by the relations $(T_i - t_i^{1/2})(T_i + t_i^{-1/2}) = 0$.

(3) Set

$$\begin{array}{c}
\begin{array}{c}
\includegraphics[width=1cm]{tikz/tikz_braid.png}
\end{array} = t_0^{1/2} \begin{array}{c}
\includegraphics[width=1cm]{tikz/tikz_braid.png}
\end{array} - \begin{array}{c}
\includegraphics[width=1cm]{tikz/tikz_braid.png}
\end{array} \quad (e_0 = t_0^{1/2} - T_0)
\end{array} \quad \begin{array}{c}
\begin{array}{c}
\includegraphics[width=1cm]{tikz/tikz_braid.png}
\end{array} = t_k^{1/2} \begin{array}{c}
\includegraphics[width=1cm]{tikz/tikz_braid.png}
\end{array} - \begin{array}{c}
\includegraphics[width=1cm]{tikz/tikz_braid.png}
\end{array} \quad (e_k = t_k^{1/2} - T_k)
\end{array} \quad \begin{array}{c}
\begin{array}{c}
\includegraphics[width=1cm]{tikz/tikz_braid.png}
\end{array} = t_i^{1/2} \begin{array}{c}
\includegraphics[width=1cm]{tikz/tikz_braid.png}
\end{array} - \begin{array}{c}
\includegraphics[width=1cm]{tikz/tikz_braid.png}
\end{array} \quad (e_i = t_i^{1/2} - T_i)
\end{array}

so that $e_j^2 = z_j e_j$ (for good $z_j$).

The two-boundary Temperley-Lieb algebra is the quotient of $H_k$ by the relations $e_i e_{i+1} e_i = e_i$ for $i = 1, \ldots, k - 1$. 
(1) The two-boundary (two-pole) braid group $B_k$ is generated by

$$T_k = \begin{array}{c}
\begin{array}{c} \circ \circ \\
\circ \circ \\
\circ \circ \\
\circ \circ \\
\end{array}
\end{array}, \quad T_0 = \begin{array}{c}
\begin{array}{c} \circ \circ \\
\circ \circ \\
\circ \circ \\
\circ \circ \\
\end{array}
\end{array} \quad \text{and} \quad T_i = \begin{array}{c}
\begin{array}{c} \circ \circ \\
\circ \circ \\
\circ \circ \\
\circ \circ \\
\end{array}
\end{array} \quad \text{for } 1 \leq i \leq k - 1.$$

(2) Fix constants $t_0, t_k, t = t_1 = t_2 = \cdots = t_{k-1} \in \mathbb{C}$.

The affine type C Hecke algebra $\mathcal{H}_k$ is the quotient of $\mathbb{C}B_k$ by the relations $(T_i - t_i^{1/2})(T_i + t_i^{-1/2}) = 0$.

(3) Set

$$= t_0^{1/2} - \begin{array}{c}
\begin{array}{c} \circ \circ \\
\circ \circ \\
\circ \circ \\
\circ \circ \\
\end{array}
\end{array}, \quad = t_k^{1/2} - \begin{array}{c}
\begin{array}{c} \circ \circ \\
\circ \circ \\
\circ \circ \\
\circ \circ \\
\end{array}
\end{array} \quad \text{and} \quad = t^{1/2} - \begin{array}{c}
\begin{array}{c} \circ \circ \\
\circ \circ \\
\circ \circ \\
\circ \circ \\
\end{array}
\end{array}$$

so that $e_j^2 = z_je_j$. The two-boundary Temperley-Lieb algebra is the quotient of $\mathcal{H}_k$ by the relations $e_ie_{i\pm1}e_i = e_i$ for $i = 1, \ldots, k - 1$. 

<table>
<thead>
<tr>
<th>Universal</th>
<th>Type B, C, D</th>
<th>Type A</th>
<th>Small Type A</th>
</tr>
</thead>
<tbody>
<tr>
<td>(orthog. &amp; sympl.)</td>
<td>(gen. &amp; sp. linear)</td>
<td>(GL$_2$ &amp; SL$_2$)</td>
<td></td>
</tr>
<tr>
<td>Two-pole braids</td>
<td>Two-pole BMW</td>
<td>Affine Hecke of type C (+twists)</td>
<td>Two-boundary TL</td>
</tr>
</tbody>
</table>
Theorem (D.-Ram)

(1) Let $U = U_q \mathfrak{g}$ for any complex reductive Lie algebras $\mathfrak{g}$. Let $M$, $N$, and $V$ be finite-dimensional modules.

The two-boundary braid group $B_k$ acts on $M \otimes (V)^{\otimes k} \otimes N$ and this action commutes with the action of $U$.

(2) If $\mathfrak{g} = \mathfrak{gl}_n$, then (for correct choices of $M$, $N$, and $V$), the affine Hecke algebra of type $C$, $H_k$, acts on $M \otimes (V)^{\otimes k} \otimes N$ and this action commutes with the action of $U$.

(3) If $\mathfrak{g} = \mathfrak{gl}_2$, then the action of the two-boundary Temperley-Lieb algebra factors through the T.L. quotient of $H_k$.
Theorem (D.-Ram)

(1) Let $U = U_q g$ for any complex reductive Lie algebras $g$. Let $M$, $N$, and $V$ be finite-dimensional modules. The two-boundary braid group $B_k$ acts on $M \otimes (V)^\otimes k \otimes N$ and this action commutes with the action of $U$.

(2) If $g = gl_n$, then (for correct choices of $M$, $N$, and $V$), the affine Hecke algebra of type $C$, $H_k$, acts on $M \otimes (V)^\otimes k \otimes N$ and this action commutes with the action of $U$.

(3) If $g = gl_2$, then the action of the two-boundary Temperley-Lieb algebra factors through the T.L. quotient of $H_k$.

Some results:

(a) A diagrammatic intuition for $H_k$.

(b) A combinatorial classification and construction of irreducible representations of $H_k$ (type C with distinct parameters) via central characters and generalizations of Young tableaux.

(c) A classification of the representations of $TL_k$ in [dGN08] via central characters, including answers to open questions and conjectures regarding their irreducibility and isomorphism classes.
Move both poles to the left.

Jucys-Murphy elements:

Pairwise commute

Laurent polynomials in \( Z \)'s

Central characters indexed by \( c \)

modulo signed permutations
Move both poles to the left

\[ M \otimes V \otimes V \otimes V \otimes V \otimes V \otimes V \otimes N \]

\[ M \otimes N \otimes V \otimes V \otimes V \otimes V \otimes V \otimes V \otimes V \]

Jucys-Murphy elements:

Pairwise commute

\[ Z_p H_k q \] is (type-C) symmetric

Laurent polynomials in \( Z_i \)’s

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\( P_C k \) (modulo signed permutations)
Move both poles to the left

Jucys-Murphy elements:

\[ Y_i = \]

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$Y_i =$

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Jucys-Murphy elements:

\[ Y_i = \]

- Pairwise commute
- \( \mathcal{Z}(\mathcal{H}_k) \) is (type-C) symmetric
- Laurent polynomials in \( Z_i \)'s
- Central characters indexed by \( c \in \mathbb{C}^k \) (modulo signed permutations)
The eigenvalues of the $T_i$’s must coincide with the eigenvalues of the corresponding $R$-matrices, which can be computed combinatorially.

$$0 = (T_0 - t_0)(T_0 - t_0^{-1}) = (T_k - t_k)(T_k - t_k^{-1}) = (T_i - t^{1/2})(T_i + t^{-1/2})$$

$$T_0 = \begin{array}{c} \bigotimes \alpha \check{R}_{VM} \check{R}_{MV} \end{array} \quad T_k = \begin{array}{c} \bigotimes \alpha \check{R}_{NV} \check{R}_{VN} \end{array} \quad T_i = \begin{array}{c} \bigotimes \alpha \check{R}_{VV} \end{array}$$
Back to tensor space operators properties

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$$0 = (T_0 - t_0)(T_0 - t_0^{-1}) = (T_k - t_k)(T_k - t_k^{-1}) = (T_i - t^{1/2})(T_i + t^{-1/2})$$

$$T_0 = \begin{array}{ccc}
        \times & \times & \times \\
        \times & \times & \times \\
        \times & \times & \times \\
        \times & \times & \times \\
        \times & \times & \times \\
        \times & \times & \times \\
    \end{array} \propto \tilde{R}_{VM} \tilde{R}_{MV} \quad T_k = \begin{array}{ccc}
        \times & \times & \times \\
        \times & \times & \times \\
        \times & \times & \times \\
        \times & \times & \times \\
        \times & \times & \times \\
        \times & \times & \times \\
    \end{array} \propto \tilde{R}_{NV} \tilde{R}_{VN} \quad T_i = \begin{array}{ccc}
        \times & \times & \times \\
        \times & \times & \times \\
        \times & \times & \times \\
        \times & \times & \times \\
        \times & \times & \times \\
        \times & \times & \times \\
    \end{array} \propto \tilde{R}_{VV}$$
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$$T_0 = \begin{array}{ll} \alpha \tilde{R}_{VM} \tilde{R}_{MV} \\ \end{array} \quad T_k = \begin{array}{ll} \alpha \tilde{R}_{NV} \tilde{R}_{VN} \\ \end{array} \quad T_i = \begin{array}{ll} \alpha \tilde{R}_{VV} \\ \end{array}$$

$$t_0 = -q^2(a_0 + b_0)$$
$$t_k = -q^2(a_k + b_k)$$
$$t = q^2$$
Exploring $M \otimes N \otimes L(\Box)^{\otimes k}$

Products of rectangles:

$$L((a^b_0)) \otimes L((a^b_k)) = \bigoplus_{\lambda \in \Lambda} L(\lambda)$$

(multiplicity one!)

where $\Lambda$ is the following set of partitions:
Exploring $\mathcal{M} \otimes \mathcal{N} \otimes L(\square)^{\otimes k}$

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Products of rectangles:

$$L((a_0^{b_0})) \otimes L((a_k^{b_k})) = \bigoplus_{\lambda \in \Lambda} L(\lambda) \quad \text{(multiplicity one!)}$$

where $\Lambda$ is the following set of partitions...
Exploring $M \otimes N \otimes L(\square)^\otimes k$

\[
\begin{array}{|c|c|}
\hline
a_0 & \\ \\
\hline
b_0 & \\ \\
\hline
\end{array}
\]

$k = 0$
Exploring $M \otimes N \otimes L(\square) \otimes k$

$k = 0$

$k = 1$
Exploring $M \otimes N \otimes L(□)\otimes k$

$k = 0$

$k = 1$
Exploring $M \otimes N \otimes L(\square)^{\otimes k}$
\[
L \left( \begin{array}{c|c|c|c|c|c|c}
1 & 2 & 3 & 4 & 5 \\
2 & 3 & 4 & 5 & 1 \\
3 & 4 & 5 & 1 & 2 \\
4 & 5 & 1 & 2 & 3 \\
5 & 1 & 2 & 3 & 4 \\
\end{array} \right) \otimes L \left( \begin{array}{c|c|c|c|c}
1 & 2 & 3 \\
2 & 3 & 1 \\
3 & 1 & 2 \\
\end{array} \right)
\]
\[ L \left( \begin{array}{cccc} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{array} \right) \otimes L \left( \begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right) \otimes L \left( \begin{array}{c} 1 \end{array} \right) \]
\[ L \left( \begin{array}{c} \hline \hline \hline \hline \hline \hline \end{array} \right) \right) \otimes L \left( \begin{array}{c} \hline \hline \hline \hline \end{array} \right) \otimes L \left( \square \right) \otimes L \left( \square \right) \]
\[
L \left( \begin{array}{c}
\vdots \\
\vdots \\
\ddots \\
\vdots \\
\vdots \\
\end{array} \right) \otimes L \left( \begin{array}{c}
\vdots \\
\vdots \\
\ddots \\
\vdots \\
\vdots \\
\end{array} \right) \otimes L (\square) \otimes L (\square) \otimes L (\square)
\]

(H) representations in tensor space are labeled by certain partitions \( \lambda \).

(B) Basis labeled by tableaux from some partition \( \mu \) in to \( \lambda \).

(C) Calibrated (\( Y_i \)'s are diagonalized): \( Y_i \) acts by \( t \) to the shifted diagonal number of box \( i \). (Think: signed permutations.)
\[ L \left( \begin{array}{c|c|c|c|c|c|c} \hline & & & & & & \\ \hline & & & & & & \\ \hline & & & & & & \\ \hline & & & & & & \\ \hline & & & & & & \\ \hline \end{array} \right) \otimes L \left( \begin{array}{c|c|c|c} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array} \right) \otimes L (\square) \otimes L (\square) \otimes L (\square) \otimes L (\square) \otimes L (\square) \]
\( L \left( \begin{array}{c|c|c|c|c} & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \end{array} \right) \right) \otimes L \left( \begin{array}{c|c|c|c|c} & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \end{array} \right) \right) \otimes L (\square) \otimes L (\square) \otimes L (\square) \otimes L (\square) \otimes L (\square) \otimes L (\square)

\((\ast)\) \(H_k\) representations in tensor space are labeled by certain partitions \(\lambda\).
$L \left( \begin{array}{c|c|c|c|c|c|c|c|c|c|c} \hline & & & & & & & & & & \\ \hline & & & & & & & & & & \\ \hline & & & & & & & & & & \\ \hline \hline & & & & & & & & & & \\ \hline & & & & & & & & & & \\ \hline & & & & & & & & & & \\ \hline \end{array} \right) \otimes L \left( \begin{array}{c|c|c|c|c|c|c|c|c|c|c} \hline & & & & & & & & & & \\ \hline & & & & & & & & & & \\ \hline & & & & & & & & & & \\ \hline \hline & & & & & & & & & & \\ \hline & & & & & & & & & & \\ \hline & & & & & & & & & & \\ \hline \end{array} \right) \otimes L (\square) \otimes L (\square) \otimes L (\square) \otimes L (\square) \otimes L (\square) \otimes L (\square)$

(*) $H_k$ representations in tensor space are labeled by certain partitions $\lambda$. 
\( L \left( \begin{array}{c|c|c|c|c|c|c|c|c|c} 
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
\end{array} \right) \otimes L \left( \begin{array}{c|c|c|c|c|c|c|c|c|c} 
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
\end{array} \right) \otimes L (\square) \otimes L (\square) \otimes L (\square) \otimes L (\square) \otimes L (\square) \otimes L (\square) \otimes L (\square) \otimes L (\square) \)
\[ L \left( \begin{array}{cccc} & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \end{array} \right) \otimes L \left( \begin{array}{cccc} & & & \\ & & & \\ & & & \\ & & & \\ & & & \end{array} \right) \otimes L \left( \begin{array}{cccc} \square & & & \\ & & & \\ & & & \\ & & & \\ & & & \end{array} \right) \otimes L \left( \begin{array}{cccc} \square & & & \\ & & & \\ & & & \\ & & & \\ & & & \end{array} \right) \otimes L \left( \begin{array}{cccc} \square & & & \\ & & & \\ & & & \\ & & & \\ & & & \end{array} \right) \otimes L \left( \begin{array}{cccc} \square & & & \\ & & & \\ & & & \\ & & & \\ & & & \end{array} \right) \otimes L \left( \begin{array}{cccc} \square & & & \\ & & & \\ & & & \\ & & & \\ & & & \end{array} \right) \]  

(*) \( H_k \) representations in tensor space are labeled by certain partitions \( \lambda \).

(*) Basis labeled by tableaux from \textit{some} partition \( \mu \) in \( (a^c) \otimes (b^d) \) to \( \lambda \).
\[ L \left( \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \\ \end{array} \right) \otimes L \left( \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \\ \end{array} \right) \otimes L (\square) \otimes L (\square) \otimes L (\square) \otimes L (\square) \otimes L (\square) \otimes L (\square) \otimes L (\square) \]

\((*)\) \(H_k\) representations in tensor space are labeled by certain partitions \(\lambda\).

\((*)\) Basis labeled by tableaux from some partition \(\mu\) in \((a^c) \otimes (b^d)\) to \(\lambda\).

\((*)\) Calibrated \((Y_i)'s\ are diagonalized)
\[ L \left( \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ \end{array} \right) \times L \left( \begin{array}{cccc} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ \end{array} \right) \times L \left( \begin{array}{cccc} 0 \\ \end{array} \right) \times L \left( \begin{array}{cccc} 0 \\ \end{array} \right) \]

(*): \( H_k \) representations in tensor space are labeled by certain partitions \( \lambda \).

(*): Basis labeled by tableaux from some partition \( \mu \) in \( (a^c) \otimes (b^d) \) to \( \lambda \).

(*): Calibrated \((Y_i)'s\) are diagonalized.
\[
L \left( \begin{array}{cccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{array} \right) \otimes L \left( \begin{array}{cc}
\vdots & \vdots \\
\vdots & \vdots \\
\end{array} \right) \otimes L \left( \begin{array}{cc}
\vdots & \vdots \\
\vdots & \vdots \\
\end{array} \right) \otimes L \left( \begin{array}{cc}
\vdots & \vdots \\
\vdots & \vdots \\
\end{array} \right) \otimes L \left( \begin{array}{cc}
\vdots & \vdots \\
\vdots & \vdots \\
\end{array} \right) \otimes L \left( \begin{array}{cc}
\vdots & \vdots \\
\vdots & \vdots \\
\end{array} \right) \otimes L \left( \begin{array}{cc}
\vdots & \vdots \\
\vdots & \vdots \\
\end{array} \right)
\]

Shift by $\frac{1}{2}(a_0 - b_0 + a_k - b_k)$

\((*)\) $H_k$ representations in tensor space are labeled by certain partitions $\lambda$.
\((*)\) Basis labeled by tableaux from some partition $\mu$ in $(a^c) \otimes (b^d)$ to $\lambda$.
\((*)\) Calibrated ($Y_i$’s are diagonalized)
\[
L \left( \begin{array}{cccc}
1 & 2 & 3 & 4 \\
5 & 6 & 7 & 8 \\
9 & 10 & 11 & 12 \\
13 & 14 & 15 & 16
\end{array} \right) \otimes L \left( \begin{array}{cccc}
1 & 2 & 3 & 4 \\
5 & 6 & 7 & 8 \\
9 & 10 & 11 & 12 \\
13 & 14 & 15 & 16
\end{array} \right) \otimes L \left( \begin{array}{cccc}
1 & 2 & 3 & 4 \\
5 & 6 & 7 & 8 \\
9 & 10 & 11 & 12 \\
13 & 14 & 15 & 16
\end{array} \right) \otimes L \left( \begin{array}{cccc}
1 & 2 & 3 & 4 \\
5 & 6 & 7 & 8 \\
9 & 10 & 11 & 12 \\
13 & 14 & 15 & 16
\end{array} \right) \otimes L \left( \begin{array}{cccc}
1 & 2 & 3 & 4 \\
5 & 6 & 7 & 8 \\
9 & 10 & 11 & 12 \\
13 & 14 & 15 & 16
\end{array} \right)
\]

Shift by \( \frac{1}{2}(a_0 - b_0 + a_k - b_k) \)

\[
Y_1 \mapsto t^{5.5} \\
Y_2 \mapsto t^{3.5} \\
Y_3 \mapsto t^{-4.5} \\
Y_4 \mapsto t^{-5.5} \\
Y_5 \mapsto t^{-2.5}
\]

\[
Y_1 \mapsto t^{-5.5} \\
Y_2 \mapsto t^{2.5} \\
Y_3 \mapsto t^{4.5} \\
Y_4 \mapsto t^{3.5} \\
Y_5 \mapsto t^{5.5}
\]

\((*)\) \(H_k\) representations in tensor space are labeled by certain partitions \(\lambda\).

\((*)\) Basis labeled by tableaux from some partition \(\mu\) in \((a^c) \otimes (b^d)\) to \(\lambda\).

\((*)\) Calibrated \((Y_i's\) are diagonalized): \(Y_i\) acts by \(t\) to the shifted diagonal number of box\(_i\).

(Think: signed permutations.)
Universal Type B, C, D (orthog. & sympl.)
Type A (gen. & sp. linear)
Small Type A (GL₂ & SL₂)

- Brauer algebra
- Sym. group
- Hecke algebra
- Temperley-Lieb

- Braid group
- BMW algebra
- Affine braids
- Affine BMW
- Two-pole braids
- Two-pole BMW

- Affine Hecke of type A (+twists)
- Affine Hecke of type C (+twists)

- One-boundary TL
- Two-boundary TL

Thanks! https://zdaugherty.ccnysites.cuny.edu/