Combinatorics and representation theory of
Temperley-Lieb algebras

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Motivating example: Schur-Weyl Duality

The symmetric group $S_k$ (permutations) as diagrams:
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The symmetric group $S_k$ (permutations) as diagrams:

(With multiplication given by concatenation)
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Motivating example: Schur-Weyl Duality

\[ \text{GL}_n(\mathbb{C}) \text{ acts on } \mathbb{C}^n \otimes \mathbb{C}^n \otimes \cdots \otimes \mathbb{C}^n = (\mathbb{C}^n)^{\otimes_k} \text{ diagonally.} \]

\[ g \cdot (v_1 \otimes v_2 \otimes \cdots \otimes v_k) = gv_1 \otimes gv_2 \otimes \cdots \otimes gv_k. \]
Motivating example: Schur-Weyl Duality

$\text{GL}_n(\mathbb{C})$ acts on $\mathbb{C}^n \otimes \mathbb{C}^n \otimes \cdots \otimes \mathbb{C}^n = (\mathbb{C}^n)^\otimes k$ diagonally.

$$g \cdot (v_1 \otimes v_2 \otimes \cdots \otimes v_k) = gv_1 \otimes gv_2 \otimes \cdots \otimes gv_k.$$  

$S_k$ also acts on $(\mathbb{C}^n)^\otimes k$ by place permutation.

![Diagram showing the action of $S_k$ on a tensor product of $\mathbb{C}^n$ vectors.]
Motivating example: Schur-Weyl Duality

$\text{GL}_n(\mathbb{C})$ acts on $\mathbb{C}^n \otimes \mathbb{C}^n \otimes \cdots \otimes \mathbb{C}^n = (\mathbb{C}^n)^\otimes k$ diagonally.

$$g \cdot (v_1 \otimes v_2 \otimes \cdots \otimes v_k) = gv_1 \otimes gv_2 \otimes \cdots \otimes gv_k.$$ 

$S_k$ also acts on $(\mathbb{C}^n)^\otimes k$ by place permutation.

These actions commute!

\[ \begin{array}{c}
\text{gv}_2 \otimes \text{gv}_4 \otimes \text{gv}_1 \otimes \text{gv}_5 \otimes \text{gv}_3 \\
\text{vs.} \\
\text{gv}_1 \otimes \text{gv}_2 \otimes \text{gv}_3 \otimes \text{gv}_4 \otimes \text{gv}_5
\end{array} \]
Motivating example: Schur-Weyl Duality

Consider the representations induced by these commuting actions,

\[ \pi : \mathbb{C}S_k \to \text{End}((\mathbb{C}^n)^\otimes k) \quad \text{and} \quad \rho : \mathbb{C}\text{GL}_n \to \text{End}((\mathbb{C}^n)^\otimes k). \]

Thm. (Schur 1901)

\[ \text{End}_{\text{GL}_n}((\mathbb{C}^n)^\otimes k) = \pi(\mathbb{C}S_k) \quad \text{and} \quad \text{End}_{S_k}((\mathbb{C}^n)^\otimes k) = \rho(\mathbb{C}\text{GL}_n). \]

(all linear maps that commute with \( \text{GL}_n \))

(img of \( S_k \) action)

(img of \( \text{GL}_n \) action)
Motivating example: Schur-Weyl Duality

Consider the representations induced by these commuting actions,
\[
\pi : \mathbb{C}S_k \rightarrow \text{End}((\mathbb{C}^n)^\otimes_k) \quad \text{and} \quad \rho : \mathbb{C}GL_n \rightarrow \text{End}((\mathbb{C}^n)^\otimes_k).
\]

Thm. (Schur 1901)

\[
\begin{align*}
\text{End}_{GL_n}((\mathbb{C}^n)^\otimes_k) &= \pi(\mathbb{C}S_k) \quad \text{and} \quad \text{End}_{S_k}((\mathbb{C}^n)^\otimes_k) = \rho(\mathbb{C}GL_n). \\
(\text{all linear maps that commute with } GL_n) & \quad (\text{img of } S_k \text{ action})
\end{align*}
\]

Powerful consequence: a duality between representations

The double-centralizer relationship produces

\[
(\mathbb{C}^n)^\otimes_k \cong \bigoplus_{\lambda \vdash k, \ell(\lambda) \leq n} G^\lambda \otimes S^\lambda \quad \text{as a } GL_n-S_k \text{ bimodule},
\]

where
\[
\begin{align*}
G^\lambda \quad &\text{are distinct irreducible } GL_n\text{-modules}, \\
S^\lambda \quad &\text{are distinct irreducible } S_k\text{-modules}.
\end{align*}
\]
Temperley-Lieb algebras

Caution! The representation

\[ \pi : \mathbb{C}S_k \to \text{End} \left( (\mathbb{C}^n)^\otimes k \right) \]

is not always injective!

**Thm.** \( \ker(\pi) \neq 0 \) when \( n < k \).
Temperley-Lieb algebras

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Thm. \( \ker(\pi) \neq 0 \) when \( n < k \).

Case \( n = 2 \): Define

\[
\begin{array}{c}
\prescript{\varepsilon}{\varepsilon} \\
\varepsilon
\end{array} = \begin{array}{c}
\varepsilon
\end{array} - \begin{array}{c}
\varepsilon
\end{array}.
\]
Temperley-Lieb algebras

Caution! The representation

$$\pi : \mathbb{C}S_k \rightarrow \text{End} \left( (\mathbb{C}^n)^\otimes k \right)$$

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Case $n = 2$: Define

$$\begin{array}{c}
\begin{array}{c}
\bullet \\
\bullet
\end{array}
= \begin{array}{c}
\bullet \\
\bullet
\end{array} - \begin{array}{c}
\bullet \\
\bullet
\end{array}.
\end{array}$$

Then in $\mathbb{C}S_k$ (for general $k$),

$$\begin{array}{c}
\begin{array}{c}
\bullet \\
\bullet
\end{array}
= \left( \begin{array}{c}
\bullet \\
\bullet
\end{array} \right)^2
\end{array}.$$
Temperley-Lieb algebras

Caution! The representation

$$\pi : \mathbb{C}S_k \to \text{End} \left( (\mathbb{C}^n)^\otimes_k \right)$$

is not always injective! [Thm. $\ker(\pi) \neq 0$ when $n < k$.]

Case $n = 2$: Define

\[
\begin{array}{c c c c c}
\begin{array}{c}
\bigcirc \bigcirc \\
\end{array} & = & \\
\begin{array}{c c c c c}
\bullet & \bullet & \bullet & \bullet \\
\end{array} & - & \\
\begin{array}{c c c c c}
\bullet & \bullet & \bullet & \bullet \\
\end{array}
\end{array}
\]

Then in $\mathbb{C}S_k$ (for general $k$),

\[
\begin{array}{c c c c c}
\begin{array}{c}
\bigcirc \bigcirc \\
\end{array} & = & \left( \begin{array}{c c c c c}
\bullet & \bullet & \bullet & \bullet \\
\end{array} \right)^2 & = & \left( \begin{array}{c c c c c}
\bullet & \bullet & \bullet & \bullet \\
\end{array} \right) \cdot \left( \begin{array}{c c c c c}
\bullet & \bullet & \bullet & \bullet \\
\end{array} \right) \cdot \left( \begin{array}{c c c c c}
\bullet & \bullet & \bullet & \bullet \\
\end{array} \right)
\end{array}
\]

Temperley-Lieb algebras

Caution! The representation

$$\pi : \mathbb{C}S_k \rightarrow \operatorname{End} \left( (\mathbb{C}^n)^\otimes k \right)$$

is not always injective!  \[ \text{Thm. } \ker(\pi) \neq 0 \text{ when } n < k. \]

Case \( n = 2 \): Define

$$\bigotimes = \bigotimes = \bigotimes.$$  

Then in \( \mathbb{C}S_k \) (for general \( k \)),

$$\bigotimes = (\bigotimes)^2 = (\bigotimes)^2 = \bigotimes + \bigotimes + \bigotimes.$$
Temperley-Lieb algebras

Caution! The representation

\[ \pi : \mathbb{C}S_k \to \text{End} \left( (\mathbb{C}^n)^\otimes k \right) \]

is not always injective! \[ \text{Thm.} \; \ker(\pi) \neq 0 \text{ when } n < k. \]

Case \( n = 2 \): Define

\[ \begin{align*}
\includegraphics[width=0.2\textwidth]{diagram1.png} &= \includegraphics[width=0.2\textwidth]{diagram2.png} - \includegraphics[width=0.2\textwidth]{diagram3.png}.
\end{align*} \]

Then in \( \mathbb{C}S_k \) (for general \( k \)),

\[ \begin{align*}
\includegraphics[width=0.2\textwidth]{diagram4.png} &= \left( \includegraphics[width=0.2\textwidth]{diagram1.png} \right)^2 = \left( \includegraphics[width=0.2\textwidth]{diagram2.png} - \includegraphics[width=0.2\textwidth]{diagram3.png} \right)^2 = \includegraphics[width=0.4\textwidth]{diagram5.png} + \includegraphics[width=0.4\textwidth]{diagram6.png} \\
&= 2 \left( \includegraphics[width=0.2\textwidth]{diagram2.png} - \includegraphics[width=0.2\textwidth]{diagram3.png} \right)
\end{align*} \]
Temperley-Lieb algebras

Caution! The representation

$$
\pi : \mathbb{C}S_k \to \text{End} \left( (\mathbb{C}^n)^\otimes k \right)
$$

is not always injective!

**Thm.** \( \ker(\pi) \neq 0 \) when \( n < k \).

Case \( n = 2 \): Define

$$
\begin{align*}
\begin{array}{c}
\includegraphics[width=2cm]{temperley_lieb_diagram1} \\
\includegraphics[width=2cm]{temperley_lieb_diagram2}
\end{array}
\end{align*}
$$

Then in \( \mathbb{C}S_k \) (for general \( k \)),

$$
\begin{align*}
\begin{array}{c}
\includegraphics[width=2cm]{temperley_lieb_diagram3} \\
\includegraphics[width=2cm]{temperley_lieb_diagram4}
\end{array}
\end{align*}
$$

Because \( \includegraphics[width=1cm]{temperley_lieb_diagram5} \) is \((2\times)\) the projection onto the sign representation for \( S_2 \).
Caution! The representation
\[ \pi : \mathbb{C}S_k \to \text{End} \left( (\mathbb{C}^n \otimes k) \right) \]
is not always injective!

**Thm.** \( \ker(\pi) \neq 0 \) when \( n < k \).

**Case** \( n = 2 \): Define
\[
\begin{array}{c}
\begin{array}{c}
\bullet \\
\bullet
\end{array}
\end{array}
= \begin{array}{c}
\begin{array}{c}
- \\
|
\end{array}
\end{array}
- \begin{array}{c}
\begin{array}{c}
\bullet \\
\bullet
\end{array}
\end{array}.
\]

*Only* true for \( n \leq 2 \):

\[
\begin{array}{c}
\begin{array}{c}
\bullet \\
\bullet
\end{array}
\end{array}
- \begin{array}{c}
\begin{array}{c}
\bullet \\
\bullet
\end{array}
\end{array} \in \ker(\pi)
\]
\[
\begin{array}{c}
\begin{array}{c}
\bullet \\
\bullet
\end{array}
\end{array}
- \begin{array}{c}
\begin{array}{c}
\bullet \\
\bullet
\end{array}
\end{array} \in \ker(\pi)
\]
Temperley-Lieb algebras

Fix $\delta \in \mathbb{C}$. The Temperley-Lieb algebra $TL_k$ is a diagram algebra generated over $\mathbb{C}$ by diagrams

$$e_i = \begin{array}{c}
\cdot \\
\cdot \\
\cdot \\
\end{array} \quad \begin{array}{c}
\cdot \\
\cdot \\
\cdot \\
\end{array} \quad \begin{array}{c}
\cdot \\
\cdot \\
\cdot \\
\end{array} \quad \begin{array}{c}
\cdot \\
\cdot \\
\cdot \\
\end{array} \quad \begin{array}{c}
\cdot \\
\cdot \\
\cdot \\
\end{array} \quad \begin{array}{c}
\cdot \\
\cdot \\
\cdot \\
\end{array}$$

for $i = 1, \ldots, k - 1$,

with relations $e_i e_j = e_j e_i$ for $|i - j| > 1$,

$$e_i e_{i \pm 1} e_i = e_i$$

for $1 \leq i \leq k - 1$,

$$e_i^2 = \delta_i e_i.$$
Temperley-Lieb algebras

Fix $\delta \in \mathbb{C}$. The Temperley-Lieb algebra $TL_k$ is a diagram algebra generated over $\mathbb{C}$ by diagrams

\[ e_i = \begin{array}{c}
\bullet \\
\vdots \\
\bullet
\end{array} \quad \cdots \quad \begin{array}{c}
\bullet \\
\begin{array}{c}
\circ \circ \\
\bullet
\end{array}
\end{array} \begin{array}{c}
\circ \circ \\
\bullet
\end{array} \cdots \begin{array}{c}
\bullet
\end{array}, \]

for $i = 1, \ldots, k - 1$,

with relations $e_i e_j = e_j e_i$ for $|i - j| > 1$,

$e_i e_{i \pm 1} e_i = e_i$

for $1 \leq i \leq k - 1$,

$e_i^2 = \delta_i e_i$.

Basis: all non-crossing diagrams

Thm. The quotient of $\mathbb{C}S_k$ by relations $(\star)$ factors through the representation $\pi: \mathbb{C}S_k \to \text{End}(\mathbb{C}^2 \otimes^k)$ (i.e. when $\delta = 2$, $TL_k$ centralizes the action of $GL_2$ on $\mathbb{C}^2 \otimes^k$).
Temperley-Lieb algebras

Fix $\delta \in \mathbb{C}$. The Temperley-Lieb algebra $TL_k$ is a diagram algebra generated over $\mathbb{C}$ by diagrams

Basis: all non-crossing diagrams

$$e_i = \begin{array}{c}
\bullet & \cdots & \bullet & \circ \circ \cdots \circ \\
\circ & \cdots & \circ & \bullet
\end{array}, \quad \text{for } i = 1, \ldots, k - 1,$$

with relations $e_i e_j = e_j e_i$ for $|i - j| > 1$,

$$e_i e_{i \pm 1} e_i = e_i$$

for $1 \leq i \leq k - 1$,

$$e_i^2 = \delta_i e_i.$$

Thm. The quotient of $\mathbb{C}S_k$ by relations ($\star$) factors through the representation

$$\pi : \mathbb{C}S_k \to \text{End} \left( (\mathbb{C}^2)^{\otimes k} \right)$$

(i.e. when $\delta = 2$, $TL_k$ centralizes the action of $\text{GL}_2$ on $(\mathbb{C}^2)^{\otimes k}$).
Quantum groups and braids

Fix \( q \in \mathbb{C} \), and let \( \mathcal{U} = \mathcal{U}_q \mathfrak{g} \) be the Drinfeld-Jimbo quantum group associated to Lie algebra \( \mathfrak{g} \) (deform the Lie algebra by a parameter \( q \)).
Quantum groups and braids

Fix $q \in \mathbb{C}$, and let $\mathcal{U} = \mathcal{U}_q \mathfrak{g}$ be the Drinfeld-Jimbo quantum group associated to Lie algebra $\mathfrak{g}$ (deform the Lie algebra by a parameter $q$). $\mathcal{U} \otimes \mathcal{U}$ has an invertible element $\mathcal{R}$ called an $R$-matrix that yields a map

$$\tilde{\mathcal{R}}_{VW} : V \otimes W \rightarrow W \otimes V$$

that (1) satisfies braid relations, and (2) commutes with the $\mathcal{U}$-action on $V \otimes W$ for any $\mathcal{U}$-module $V$. 
Quantum groups and braids

Fix \( q \in \mathbb{C} \), and let \( \mathcal{U} = \mathcal{U}_q \mathfrak{g} \) be the Drinfeld-Jimbo quantum group associated to Lie algebra \( \mathfrak{g} \) (deform the Lie algebra by a parameter \( q \)). \( \mathcal{U} \otimes \mathcal{U} \) has an invertible element \( \mathcal{R} \) called an R-matrix that yields a map

\[
\tilde{\mathcal{R}}_{vw} : V \otimes W \longrightarrow W \otimes V
\]

that

1. satisfies braid relations, and
2. commutes with the \( \mathcal{U} \)-action on \( V \otimes W \)

for any \( \mathcal{U} \)-module \( V \).

The braid group \( \mathcal{B}_k \) shares a commuting action with \( \mathcal{U} \) on \( V^\otimes k \):
Quantum groups and braids

Fix $q \in \mathbb{C}$, and let $\mathcal{U} = \mathcal{U}_q \mathfrak{g}$ be the Drinfeld-Jimbo quantum group associated to Lie algebra $\mathfrak{g}$ (deform the Lie algebra by a parameter $q$). $\mathcal{U} \otimes \mathcal{U}$ has an invertible element $\mathcal{R}$ called an $R$-matrix that yields a map

$$\mathcal{R}_{VW} : V \otimes W \rightarrow W \otimes V$$

that (1) satisfies braid relations, and (2) commutes with the $\mathcal{U}$-action on $V \otimes W$ for any $\mathcal{U}$-module $V$.

The one-pole/affine braid group $\mathcal{B}^{(1)}_k$ shares a commuting action with $\mathcal{U}$ on $M \otimes V^\otimes k$:

Around the pole: $\mathcal{R}_{MV} \mathcal{R}_{VM}$
Quantum groups and braids

Fix $q \in \mathbb{C}$, and let $\mathcal{U} = \mathcal{U}_q g$ be the Drinfeld-Jimbo quantum group associated to Lie algebra $g$ (deform the Lie algebra by a parameter $q$). $\mathcal{U} \otimes \mathcal{U}$ has an invertible element $\mathcal{R}$ called an R-matrix that yields a map

$$\tilde{\mathcal{R}}_{VW} : V \otimes W \rightarrow W \otimes V$$

that

1. satisfies braid relations, and
2. commutes with the $\mathcal{U}$-action on $V \otimes W$

for any $\mathcal{U}$-module $V$.

The two-pole braid group $B^{(2)}_k$ shares a commuting action with $\mathcal{U}$ on $M \otimes V \otimes^k \otimes N$:

Around the pole:

$$= \tilde{\mathcal{R}}_{MV} \tilde{\mathcal{R}}_{VM}$$
The type-A Hecke algebra is the quotient of the group algebra of the braid group $\mathcal{B}_k$ by relations

\[
\begin{array}{c}
\bullet \bullet \\
\end{array} = (q - q^{-1}) \begin{array}{c}
\bullet \\
\end{array} + \begin{array}{c}
\bullet \\
\end{array}.
\]

(\ast)

**Thm.** The action of $\mathbb{C}\mathcal{B}_k$ on $V \otimes^k$ factors through the quotient by (\ast) when $V = \mathbb{C}^n$ and $\mathfrak{g} = \mathfrak{gl}_n$ or $\mathfrak{sl}_n$. 
The type-A Hecke algebra is the quotient of the group algebra of the braid group $B_k$ by relations

$$\begin{align*}
\begin{array}{c}
\text{\includegraphics[width=2cm]{type-A-hecke-diagram}}
\end{array}
= (q - q^{-1}) \begin{array}{c}
\text{\includegraphics[width=2cm]{type-A-hecke-diagram}}
\end{array} + \begin{array}{c}
\text{\includegraphics[width=2cm]{type-A-hecke-diagram}}
\end{array}. 
\end{align*}
\tag{*}$$

Thm. The action of $\mathbb{C}B_k$ on $V^\otimes k$ factors through the quotient by (*) when $V = \mathbb{C}^n$ and $\mathfrak{g} = \mathfrak{gl}_n$ or $\mathfrak{sl}_n$.

The affine type-$GL_k$ Hecke algebra is the quotient of the group algebra of the one-pole braid group $B_k^{(1)}$ by relations (*).

Thm. The action of $\mathbb{C}B_k^{(1)}$ on $M \otimes V^\otimes k$ factors through the quotient by (*) when $V = \mathbb{C}^n$ and $\mathfrak{g} = \mathfrak{gl}_n$ or $\mathfrak{sl}_n$. 

---

Type A

Affine Type

Affine Type C

Dynkin diagrams:
The type-A Hecke algebra is the quotient of the group algebra of the braid group $B_k$ by relations

\[ \begin{array}{c}
\raisebox{-1cm}{\includegraphics[width=0.5\textwidth]{braid_diagram}} \\
\end{array} = (q - q^{-1}) \begin{array}{c}
\raisebox{-1cm}{\includegraphics[width=0.5\textwidth]{braid_diagram}} \\
\end{array} + 
\]

\[ (*) \]

Thm. The action of $\mathbb{C}B_k$ on $V \otimes^k$ factors through the quotient by $(*)$ when $V = \mathbb{C}^n$ and $\mathfrak{g} = \mathfrak{gl}_n$ or $\mathfrak{sl}_n$.

The affine type-$GL_k$ Hecke algebra is the quotient of the group algebra of the one-pole braid group $B_k^{(1)}$ by relations $(*)$.

Thm. The action of $\mathbb{C}B_k^{(1)}$ on $M \otimes V \otimes^k$ factors through the quotient by $(*)$ when $V = \mathbb{C}^n$ and $\mathfrak{g} = \mathfrak{gl}_n$ or $\mathfrak{sl}_n$.

The affine type-C Hecke algebra is the quotient of the group algebra of the two-pole braid group $B_k^{(2)}$ by relations $(*)$,

\[ \begin{array}{c}
\begin{array}{c}
\raisebox{-1cm}{\includegraphics[width=0.5\textwidth]{braid_diagram}} \\
\end{array} = a \begin{array}{c}
\begin{array}{c}
\raisebox{-1cm}{\includegraphics[width=0.5\textwidth]{braid_diagram}} \\
\end{array} \\
\end{array} \\
\end{array} + 
\]

\[ (***) \]

Thm. The action of $\mathbb{C}B_k^{(2)}$ on $M \otimes V \otimes^k \otimes N$ factors through the quotient by $(*)$ and $(***)$ when $V = \mathbb{C}^n$, $M$ and $N$ are “rectangular”, and $\mathfrak{g} = \mathfrak{gl}_n$ or $\mathfrak{sl}_n$. 

Dynkin diagrams:

···

Type A
···

Affine Type
···

Affine Type GL
···

Affine Type C
The type-A Hecke algebra is the quotient of the group algebra of the braid group $\mathcal{B}_k$ by relations

\[
\begin{array}{c}
\begin{array}{c}
\text{Diagram 1}
\end{array}
\end{array}
= (q - q^{-1}) \begin{array}{c}
\begin{array}{c}
\text{Diagram 2}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\text{Diagram 3}
\end{array}
\end{array}.
\]

The affine type-$GL_k$ Hecke algebra is the quotient of the group algebra of the one-pole braid group $\mathcal{B}_k^{(1)}$ by relations ($\ast$).

The affine type-C Hecke algebra is the quotient of the group algebra of the two-pole braid group $\mathcal{B}_k^{(2)}$ by relations ($\ast$),

\[
\begin{array}{c}
\begin{array}{c}
\text{Diagram 4}
\end{array}
\end{array}
= a \begin{array}{c}
\begin{array}{c}
\text{Diagram 5}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\text{Diagram 6}
\end{array}
\end{array} \quad \text{and} \quad \begin{array}{c}
\begin{array}{c}
\text{Diagram 7}
\end{array}
\end{array}
= b \begin{array}{c}
\begin{array}{c}
\text{Diagram 8}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\text{Diagram 9}
\end{array}
\end{array} \]

\[
\]

Type what-now?
Dynkin diagrams:

Type A

Affine Type $GL$

Affine Type C

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```

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```
The type-A Hecke algebra is the quotient of the group algebra of the braid group $B_k$ by relations

$$
\begin{array}{c}
\begin{array}{c}
\includegraphics{typeA_diagram.png}
\end{array}
\end{array}
= (q - q^{-1}) \begin{array}{c}
\begin{array}{c}
\includegraphics{typeA_diagram.png}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\includegraphics{typeA_diagram.png}
\end{array}
\end{array}.
\end{array}
\quad (\ast)

The affine type-GL$_k$ Hecke algebra is the quotient of the group algebra of the one-pole braid group $B_k^{(1)}$ by relations ($\ast$).

The affine type-C Hecke algebra is the quotient of the group algebra of the two-pole braid group $B_k^{(2)}$ by relations ($\ast$),

$$
\begin{array}{c}
\begin{array}{c}
\includegraphics{typeC_diagram.png}
\end{array}
\end{array}
= a \begin{array}{c}
\begin{array}{c}
\includegraphics{typeC_diagram.png}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\includegraphics{typeC_diagram.png}
\end{array}
\end{array} \quad \text{and} \quad 
\begin{array}{c}
\begin{array}{c}
\includegraphics{typeC_diagram.png}
\end{array}
\end{array}
= b \begin{array}{c}
\begin{array}{c}
\includegraphics{typeC_diagram.png}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\includegraphics{typeC_diagram.png}
\end{array}
\end{array}
\end{array}
\quad (\ast\ast)

“Type what-now?”

Dynkin diagrams:

<table>
<thead>
<tr>
<th>Type A</th>
<th>Affine Type GL</th>
<th>Affine Type C</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\includegraphics{typeA_diagram.png}$</td>
<td>$\includegraphics{affineTypeGL_diagram.png}$</td>
<td>$\includegraphics{affineTypeC_diagram.png}$</td>
</tr>
</tbody>
</table>
The two-pole/affine type-C braid group is the group $B_k^{(2)}$ generated by $T_0, T_1, \ldots, T_k$, with relations

$$T_0 T_1 T_2 \cdots T_{k-2} T_{k-1} T_k$$

Pictorially, the generators of $B_k^{(2)}$ are identified with the diagrams

\[ T_0 \quad T_1 \quad T_2 \quad \cdots \quad T_{k-2} \quad T_{k-1} \quad T_k \]
The two-pole/affine type-C braid group is the group $\mathcal{B}_k^{(2)}$ generated by $T_0, T_1, \ldots, T_k$, with relations

\[
\begin{array}{cccccc}
T_0 & T_1 & T_2 & \cdots & T_{k-2} & T_{k-1} & T_k \\
\includegraphics{diagram1} & \includegraphics{diagram2} & \includegraphics{diagram3} & \cdots & \includegraphics{diagram4} & \includegraphics{diagram5} & \includegraphics{diagram6}
\end{array}
\]

Pictorially, the generators of $\mathcal{B}_k^{(2)}$ are identified with the diagrams

\[
T_k = \begin{array}{cccccccc}
\includegraphics{diagram7} \\
\includegraphics{diagram8}
\end{array}, \quad T_0 = \begin{array}{cccccccc}
\includegraphics{diagram9} \\
\includegraphics{diagram10}
\end{array},
\]

and

\[
T_i = \begin{array}{cccccccc}
\includegraphics{diagram11}
\end{array} \quad \text{for } i = 1, \ldots, k - 1.
\]
The two-pole/affine type-C braid group is the group $B_{k}^{(2)}$ generated by $T_0, T_1, \ldots, T_k$, with relations

\[
T_0 T_1 T_2 \cdots T_{k-2} T_{k-1} T_k
\]

Pictorially,

\[
T_i T_{i+1} T_i = \text{[diagram]} = \text{[diagram]} = T_{i+1} T_i T_{i+1}
\]
The two-pole/affine type-C braid group is the group $B_k^{(2)}$ generated by $T_0, T_1, \ldots, T_k$, with relations

$$T_0 T_1 T_2 \cdots T_{k-2} T_{k-1} T_k = T_k T_{k-1} T_k T_{k-2} \cdots T_1 T_0 = T_{k+1} T_k T_{k-1} T_0 T_1 T_2 \cdots T_{k-2} T_{k-1} T_k = T_{k-1} T_k T_{k+1} T_k T_{k-1} T_0 T_1 T_2 \cdots T_{k-2} T_{k-1} T_k$$

Pictorially,

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$$

(similar picture for $T_k T_{k-1} T_k T_{k-1} = T_{k-1} T_k T_{k-1} T_k$)
Back to Temperley-Lieb algebras

The type-A Hecke algebra $HA_k$ is the quotient of the group algebra of the braid group $B_k$ by relations

\[
\begin{array}{c}
\text{\includegraphics{hecke_relation.png}} \\
= (q - q^{-1}) \text{\includegraphics{hecke_relation.png}} + \text{\includegraphics{identity.png}}.
\end{array}
\]  

\textbf{Thm.} The action of $\mathbb{C}B_k$ on $V^\otimes k$ factors through the quotient by (*) when $V = \mathbb{C}^n$ and $\mathfrak{g} = \mathfrak{gl}_n$ or $\mathfrak{sl}_n$.
Back to Temperley-Lieb algebras

The type-A Hecke algebra $H A_k$ is the quotient of the group algebra of the braid group $B_k$ by relations

$$
\begin{align*}
&= (q - q^{-1}) \begin{array}{c}
\text{braids}
\end{array} + \begin{array}{c}
\text{skein diagram}
\end{array}.
\end{align*}
$$

(\ast)

Thm. The action of $\mathbb{C} B_k$ on $V \otimes^k$ factors through the quotient by (\ast) when $V = \mathbb{C}^n$ and $\mathfrak{g} = \mathfrak{gl}_n$ or $\mathfrak{sl}_n$.

Case $n = 2$: Define

$$
\begin{align*}
&= q \begin{array}{c}
\text{braids}
\end{array} - \begin{array}{c}
\text{skein diagram}
\end{array}.
\end{align*}
$$

(\lozenge)
Back to Temperley-Lieb algebras

The type-A Hecke algebra $HA_k$ is the quotient of the group algebra of the braid group $B_k$ by relations

$$= (q - q^{-1}) +.$$

\[ (*) \]

Thm. The action of $\mathbb{C}B_k$ on $V \otimes^k$ factors through the quotient by $(*)$ when $V = \mathbb{C}^n$ and $g = \mathfrak{gl}_n$ or $\mathfrak{sl}_n$.

Case $n = 2$: Define

$$= q -.$$

\[ (\diamond) \]

Then in $HA_k$ (for general $k$),

$$= \left( \right)^2.$$
Back to Temperley-Lieb algebras

The type-A Hecke algebra $HA_k$ is the quotient of the group algebra of the braid group $B_k$ by relations

$$
\begin{array}{c}
\begin{array}{c}
\quad
\end{array}
\end{array}
= (q - q^{-1})
\begin{array}{c}
\begin{array}{c}
\quad
\end{array}
\end{array}
+ \begin{array}{c}
\begin{array}{c}
\quad
\end{array}
\end{array}.
\tag{\star}

\textbf{Thm.} The action of $\mathbb{C}B_k$ on $V^\otimes k$ factors through the quotient by (\star) when $V = \mathbb{C}^n$ and $g = \mathfrak{gl}_n$ or $\mathfrak{sl}_n$.

\textbf{Case } n = 2: \text{ Define}

$$
\begin{array}{c}
\begin{array}{c}
\quad
\end{array}
\end{array}
= q
\begin{array}{c}
\begin{array}{c}
\quad
\end{array}
\end{array}
- \begin{array}{c}
\begin{array}{c}
\quad
\end{array}
\end{array}.
\tag{\diamond}

\text{Then in } HA_k \text{ (for general } k),

$$
\begin{array}{c}
\begin{array}{c}
\quad
\end{array}
\end{array}
= \left(\begin{array}{c}
\begin{array}{c}
\quad
\end{array}
\end{array}\right)^2
= \left(q
\begin{array}{c}
\begin{array}{c}
\quad
\end{array}
\end{array}
- \begin{array}{c}
\begin{array}{c}
\quad
\end{array}
\end{array}\right)^2.
Back to Temperley-Lieb algebras

The type-A Hecke algebra $HA_k$ is the quotient of the group algebra of the braid group $B_k$ by relations

$$
\raisebox{-0.5em}{\begin{tikzpicture}
  
  
  \draw (-0.5,-0.5) -- (0.5,0.5);
  \filldraw (0,0) circle (2pt);
  \draw (0,0) -- (0,1);
  \filldraw (0,1) circle (2pt);
  \draw (-0.5,0) -- (-0.5,1);
  \filldraw (-0.5,1) circle (2pt);
  \draw (0,1) -- (0.5,0);
  \filldraw (0.5,0) circle (2pt);
  \draw (-0.5,0) -- (-0.5,-0.5);
  \filldraw (-0.5,-0.5) circle (2pt);
  \draw (0,-0.5) -- (0,0);
  \filldraw (0,0) circle (2pt);
  \end{tikzpicture}} = (q - q^{-1}) \raisebox{-0.5em}{\begin{tikzpicture}
  
  \draw (-0.5,-0.5) -- (0.5,0.5);
  \filldraw (0,0) circle (2pt);
  \draw (0,0) -- (0,1);
  \filldraw (0,1) circle (2pt);
  \draw (-0.5,0) -- (-0.5,1);
  \filldraw (-0.5,1) circle (2pt);
  \draw (0,1) -- (0.5,0);
  \filldraw (0.5,0) circle (2pt);
  \draw (-0.5,0) -- (-0.5,-0.5);
  \filldraw (-0.5,-0.5) circle (2pt);
  \draw (0,-0.5) -- (0,0);
  \filldraw (0,0) circle (2pt);
  \end{tikzpicture}} + \raisebox{-0.5em}{\begin{tikzpicture}
  
  \filldraw (0,0) circle (2pt);
  \filldraw (0,1) circle (2pt);
  \filldraw (-0.5,0) circle (2pt);
  \filldraw (-0.5,1) circle (2pt);
  \filldraw (0.5,0) circle (2pt);
  \filldraw (0.5,1) circle (2pt);
  \end{tikzpicture}}. \quad (*)
$$

**Thm.** The action of $\mathbb{C}B_k$ on $V^\otimes k$ factors through the quotient by $(*)$ when $V = \mathbb{C}^n$ and $g = gl_n$ or $sl_n$.

**Case $n = 2$:** Define

$$
\raisebox{-0.5em}{\begin{tikzpicture}
  
  \draw (-0.5,-0.5) -- (0.5,0.5);
  \filldraw (0,0) circle (2pt);
  \draw (0,0) -- (0,1);
  \filldraw (0,1) circle (2pt);
  \draw (-0.5,0) -- (-0.5,1);
  \filldraw (-0.5,1) circle (2pt);
  \draw (0,1) -- (0.5,0);
  \filldraw (0.5,0) circle (2pt);
  \draw (-0.5,0) -- (-0.5,-0.5);
  \filldraw (-0.5,-0.5) circle (2pt);
  \draw (0,-0.5) -- (0,0);
  \filldraw (0,0) circle (2pt);
  \end{tikzpicture}} = q \raisebox{-0.5em}{\begin{tikzpicture}
  
  \filldraw (0,0) circle (2pt);
  \filldraw (0,1) circle (2pt);
  \filldraw (-0.5,0) circle (2pt);
  \filldraw (-0.5,1) circle (2pt);
  \filldraw (0.5,0) circle (2pt);
  \filldraw (0.5,1) circle (2pt);
  \end{tikzpicture}} - \raisebox{-0.5em}{\begin{tikzpicture}
  
  \draw (-0.5,-0.5) -- (0.5,0.5);
  \filldraw (0,0) circle (2pt);
  \draw (0,0) -- (0,1);
  \filldraw (0,1) circle (2pt);
  \draw (-0.5,0) -- (-0.5,1);
  \filldraw (-0.5,1) circle (2pt);
  \draw (0,1) -- (0.5,0);
  \filldraw (0.5,0) circle (2pt);
  \draw (-0.5,0) -- (-0.5,-0.5);
  \filldraw (-0.5,-0.5) circle (2pt);
  \draw (0,-0.5) -- (0,0);
  \filldraw (0,0) circle (2pt);
  \end{tikzpicture}}. \quad (\diamond)
$$

Then in $HA_k$ (for general $k$),

$$
\raisebox{-0.5em}{\begin{tikzpicture}
  
  \filldraw (0,0) circle (2pt);
  \filldraw (0,1) circle (2pt);
  \filldraw (-0.5,0) circle (2pt);
  \filldraw (-0.5,1) circle (2pt);
  \filldraw (0.5,0) circle (2pt);
  \filldraw (0.5,1) circle (2pt);
  \end{tikzpicture}} = \left(\raisebox{-0.5em}{\begin{tikzpicture}
  
  \draw (-0.5,-0.5) -- (0.5,0.5);
  \filldraw (0,0) circle (2pt);
  \draw (0,0) -- (0,1);
  \filldraw (0,1) circle (2pt);
  \draw (-0.5,0) -- (-0.5,1);
  \filldraw (-0.5,1) circle (2pt);
  \draw (0,1) -- (0.5,0);
  \filldraw (0.5,0) circle (2pt);
  \draw (-0.5,0) -- (-0.5,-0.5);
  \filldraw (-0.5,-0.5) circle (2pt);
  \draw (0,-0.5) -- (0,0);
  \filldraw (0,0) circle (2pt);
  \end{tikzpicture}}\right)^2 = \left(q \raisebox{-0.5em}{\begin{tikzpicture}
  
  \filldraw (0,0) circle (2pt);
  \filldraw (0,1) circle (2pt);
  \filldraw (-0.5,0) circle (2pt);
  \filldraw (-0.5,1) circle (2pt);
  \filldraw (0.5,0) circle (2pt);
  \filldraw (0.5,1) circle (2pt);
  \end{tikzpicture}} - \raisebox{-0.5em}{\begin{tikzpicture}
  
  \draw (-0.5,-0.5) -- (0.5,0.5);
  \filldraw (0,0) circle (2pt);
  \draw (0,0) -- (0,1);
  \filldraw (0,1) circle (2pt);
  \draw (-0.5,0) -- (-0.5,1);
  \filldraw (-0.5,1) circle (2pt);
  \draw (0,1) -- (0.5,0);
  \filldraw (0.5,0) circle (2pt);
  \draw (-0.5,0) -- (-0.5,-0.5);
  \filldraw (-0.5,-0.5) circle (2pt);
  \draw (0,-0.5) -- (0,0);
  \filldraw (0,0) circle (2pt);
  \end{tikzpicture}}\right)^2 = q^2 \raisebox{-0.5em}{\begin{tikzpicture}
  
  \filldraw (0,0) circle (2pt);
  \filldraw (0,1) circle (2pt);
  \filldraw (-0.5,0) circle (2pt);
  \filldraw (-0.5,1) circle (2pt);
  \filldraw (0.5,0) circle (2pt);
  \filldraw (0.5,1) circle (2pt);
  \end{tikzpicture}} - q \raisebox{-0.5em}{\begin{tikzpicture}
  
  \draw (-0.5,-0.5) -- (0.5,0.5);
  \filldraw (0,0) circle (2pt);
  \draw (0,0) -- (0,1);
  \filldraw (0,1) circle (2pt);
  \draw (-0.5,0) -- (-0.5,1);
  \filldraw (-0.5,1) circle (2pt);
  \draw (0,1) -- (0.5,0);
  \filldraw (0.5,0) circle (2pt);
  \draw (-0.5,0) -- (-0.5,-0.5);
  \filldraw (-0.5,-0.5) circle (2pt);
  \draw (0,-0.5) -- (0,0);
  \filldraw (0,0) circle (2pt);
  \end{tikzpicture}} - q \raisebox{-0.5em}{\begin{tikzpicture}
  
  \draw (-0.5,-0.5) -- (0.5,0.5);
  \filldraw (0,0) circle (2pt);
  \draw (0,0) -- (0,1);
  \filldraw (0,1) circle (2pt);
  \draw (-0.5,0) -- (-0.5,1);
  \filldraw (-0.5,1) circle (2pt);
  \draw (0,1) -- (0.5,0);
  \filldraw (0.5,0) circle (2pt);
  \draw (-0.5,0) -- (-0.5,-0.5);
  \filldraw (-0.5,-0.5) circle (2pt);
  \draw (0,-0.5) -- (0,0);
  \filldraw (0,0) circle (2pt);
  \end{tikzpicture}} + \raisebox{-0.5em}{\begin{tikzpicture}
  
  \filldraw (0,0) circle (2pt);
  \filldraw (0,1) circle (2pt);
  \filldraw (-0.5,0) circle (2pt);
  \filldraw (-0.5,1) circle (2pt);
  \filldraw (0.5,0) circle (2pt);
  \filldraw (0.5,1) circle (2pt);
  \end{tikzpicture}}.$$
Back to Temperley-Lieb algebras

The type-A Hecke algebra $HA_k$ is the quotient of the group algebra of the braid group $B_k$ by relations

$$
\begin{array}{c}
\begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\end{array}
\end{array}
= (q - q^{-1})
\begin{array}{c}
\begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\end{array}
\end{array}
+ \begin{array}{c}
\begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\end{array}
\end{array}.
\end{array}
\quad (*)
$$

Thm. The action of $CB_k$ on $V^\otimes k$ factors through the quotient by $(*)$ when $V = \mathbb{C}^n$ and $g = gl_n$ or $sl_n$.

Case $n = 2$: Define

$$
\begin{array}{c}
\begin{array}{c}
\bullet \\
\bullet \\
\end{array}
\end{array}
= q
\begin{array}{c}
\begin{array}{c}
\bullet \\
\bullet \\
\end{array}
\end{array}
- \begin{array}{c}
\begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\end{array}
\end{array}.
\end{array}
\quad (\Diamond)
$$

Then in $HA_k$ (for general $k$),

$$
\begin{array}{c}
\begin{array}{c}
\bullet \\
\bullet \\
\end{array}
\end{array}
= \left(\begin{array}{c}
\begin{array}{c}
\bullet \\
\bullet \\
\end{array}
\end{array}\right)^2
= \left(\begin{array}{c}
\begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\end{array}
\end{array}\right)^2
= q^2
\begin{array}{c}
\begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\end{array}
\end{array}
- q
\begin{array}{c}
\begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\end{array}
\end{array}
- q
\begin{array}{c}
\begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\end{array}
\end{array}
+ 
\begin{array}{c}
\begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\end{array}
\end{array}
= q^2
\begin{array}{c}
\begin{array}{c}
\bullet \\
\bullet \\
\end{array}
\end{array}
- q
\begin{array}{c}
\begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\end{array}
\end{array}
+ \left(\begin{array}{c}
\begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\end{array}
\end{array}\right) + \left(\begin{array}{c}
\begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\end{array}
\end{array}\right).
\end{array}
$$
Back to Temperley-Lieb algebras

The type-A Hecke algebra $HA_k$ is the quotient of the group algebra of the braid group $B_k$ by relations

$$
\begin{array}{c}
\begin{array}{c}
\text{Diagram 1}
\end{array}
\end{array}
= (q - q^{-1}) \begin{array}{c}
\begin{array}{c}
\text{Diagram 2}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\text{Diagram 3}
\end{array}
\end{array}.
\end{array}
(\ast)

Thm. The action of $\mathbb{C}B_k$ on $V^\otimes k$ factors through the quotient by $(\ast)$ when $V = \mathbb{C}^n$ and $g = \mathfrak{gl}_n$ or $\mathfrak{sl}_n$.

Case $n = 2$: Define

$$
\begin{array}{c}
\begin{array}{c}
\text{Diagram 4}
\end{array}
\end{array}
= q \begin{array}{c}
\begin{array}{c}
\text{Diagram 5}
\end{array}
\end{array} - \begin{array}{c}
\begin{array}{c}
\text{Diagram 6}
\end{array}
\end{array}.
\end{array}
(\diamondsuit)

Then in $HA_k$ (for general $k$),

$$
\begin{array}{c}
\begin{array}{c}
\text{Diagram 7}
\end{array}
\end{array}
= \left(\begin{array}{c}
\begin{array}{c}
\text{Diagram 8}
\end{array}
\end{array}\right)^2 = \left(\begin{array}{c}
\begin{array}{c}
\text{Diagram 9}
\end{array}
\end{array}\right)^2 = q^2 \begin{array}{c}
\begin{array}{c}
\text{Diagram 10}
\end{array}
\end{array} - q \begin{array}{c}
\begin{array}{c}
\text{Diagram 11}
\end{array}
\end{array} - q \begin{array}{c}
\begin{array}{c}
\text{Diagram 12}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\text{Diagram 13}
\end{array}
\end{array}
\end{array}

= q^2 \begin{array}{c}
\begin{array}{c}
\text{Diagram 14}
\end{array}
\end{array} - q \begin{array}{c}
\begin{array}{c}
\text{Diagram 15}
\end{array}
\end{array} + \left(\begin{array}{c}
\begin{array}{c}
\text{Diagram 16}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\text{Diagram 17}
\end{array}
\end{array}\right) = (q + q^{-1}) \begin{array}{c}
\begin{array}{c}
\text{Diagram 18}
\end{array}
\end{array}.
\end{array}

Because $\begin{array}{c}
\begin{array}{c}
\text{Diagram 19}
\end{array}
\end{array}$ is $(q + q^{-1}) \times \text{(proj. onto sign representation for } HA_2\text{)}$. 
The type-A Hecke algebra $HA_k$ is the quotient of the group algebra of the braid group $B_k$ by relations

\[ \begin{array}{c}
\begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet
\end{array}
\end{array} = (q - q^{-1}) \begin{array}{c}
\begin{array}{c}
\bullet \\
\bullet
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\bullet
\end{array}
\end{array}. \tag{\ast}
\]

**Thm.** The action of $\mathbb{C}B_k$ on $V \otimes^k$ factors through the quotient by $(\ast)$ when $V = \mathbb{C}^n$ and $\mathfrak{g} = \mathfrak{gl}_n$ or $\mathfrak{sl}_n$.

---

**Case $n = 2$:** Define

\[ \begin{array}{c}
\begin{array}{c}
\bullet \\
\bullet
\end{array}
\end{array} = q \begin{array}{c}
\begin{array}{c}
\bullet
\end{array}
\end{array} - \begin{array}{c}
\begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet
\end{array}
\end{array}. \tag{\diamond}
\]

**Thm.** Using the identification in $(\diamond)$, the action of $HA_k$ on $(\mathbb{C}^2) \otimes^k$ factors through the Temperley-Lieb quotient when $\delta = q + q^{-1} = [2]_q$, i.e. $TL_k$ centralizes $U_q\mathfrak{sl}_2$ and $U_q\mathfrak{sl}_2$ in $\text{End}((\mathbb{C}^2) \otimes^k)$ when $\bigodot = [2]_q$. 
\[
\begin{align*}
\begin{array}{c}
\text{tensor space} \\
\text{centralizer of } \mathcal{U}_q g \\
V \otimes k \\
M \otimes V \otimes k \otimes N
\end{array}
\end{align*}
\]

<table>
<thead>
<tr>
<th>tensor space</th>
<th>centralizer of ( \mathcal{U}_q g )</th>
<th>centralizer of ( \mathcal{U}_q \mathfrak{gl}_n )</th>
<th>centralizer of ( \mathcal{U}_q \mathfrak{gl}_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( V \otimes k )</td>
<td>Braids on ( k ) strands</td>
<td>Type-A Hecke (twist relations)</td>
<td>Temperley-Lieb (twist relations)</td>
</tr>
<tr>
<td>( M \otimes V \otimes k )</td>
<td>One-pole braids</td>
<td>Affine type-GL Hecke (twist relations)</td>
<td>1-boundary TL</td>
</tr>
<tr>
<td>( M \otimes V \otimes k \otimes N )</td>
<td>Two-pole braids</td>
<td>Affine type-C Hecke (twist &amp; wrap relations)</td>
<td>2-boundary TL</td>
</tr>
</tbody>
</table>

Diagram:
- Braids on \( k \) strands
- Type-A Hecke
- Temperley-Lieb
- Affine type-GL Hecke
- 1-boundary TL
- Affine type-C Hecke
- 2-boundary TL
Two-boundary Temperley-Lieb algebras

[MNGB04] Fix $\delta, \delta_0, \delta_k \in \mathbb{C}$. The two-boundary Temperley-Lieb algebra $TL^{(2)}_k$ is a diagram algebra generated over $\mathbb{C}$ by diagrams

$$e_0 = \begin{array}{c}
1 \\
1
\end{array}, \quad e_k = \begin{array}{c}
1 \\
1
\end{array}, \quad \text{and} \quad e_i = \begin{array}{c}
1 \\
1
\end{array},$$

for $i = 1, \ldots, k - 1$
Two-boundary Temperley-Lieb algebras

[MNGB04] Fix $\delta, \delta_0, \delta_k \in \mathbb{C}$. The two-boundary Temperley-Lieb algebra $TL^{(2)}_k$ is a diagram algebra generated over $\mathbb{C}$ by diagrams

\[ e_0 = \begin{array}{c}
\includegraphics{fig1}
\end{array}, \quad e_k = \begin{array}{c}
\includegraphics{fig2}
\end{array}, \quad \text{and} \quad e_i = \begin{array}{c}
\includegraphics{fig3}
\end{array} \]

for $i = 1, \ldots, k - 1$, with relations $e_i e_j = e_j e_i$ for $|i - j| > 1$,

\[ e_i e_{i \pm 1} e_i = e_i \]

for $1 \leq i \leq k - 1$,

\[ e_i^2 = \delta_i e_i. \]
Two-boundary Temperley-Lieb algebras

[MNGB04] Fix $\delta, \delta_0, \delta_k \in \mathbb{C}$. The two-boundary Temperley-Lieb algebra $TL^{(2)}_k$ is a diagram algebra generated over $\mathbb{C}$ by diagrams

$$e_0 = \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1
\end{array}
\end{array}
\end{array},$$

$$e_k = \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
k
\end{array}
\end{array}
\end{array},$$

and

$$e_i = \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
i
\end{array}
\end{array}
\end{array}$$

for $i = 1, \ldots, k - 1$, with relations $e_i e_j = e_j e_i$ for $|i - j| > 1$,

$$e_i e_{i \pm 1} e_i = e_i$$
for $1 \leq i \leq k - 1$,

$$e_i^2 = \delta_i e_i.$$
Two-boundary Temperley-Lieb algebras

[MNGB04] Fix $\delta, \delta_0, \delta_k \in \mathbb{C}$. The two-boundary Temperley-Lieb algebra $TL_k^{(2)}$ is a diagram algebra generated over $\mathbb{C}$ by diagrams

$e_0 = \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1 \quad 1
\end{array}
\end{array}
\end{array}
\end{array}$,

$e_k = \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
k

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\end{array}$,

and $e_i = \begin{array}{c}
\begin{array}{c}
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\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
i

\end{array}
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\end{array}
\end{array}
\end{array}
\end{array}$

for $i = 1, \ldots, k - 1$, with relations $e_i e_j = e_j e_i$ for $|i - j| > 1$,

$e_i e_{i \pm 1} e_i = e_i$

for $1 \leq i \leq k - 1$,

$e_i^2 = \delta_i e_i$.

or

or

or

or

or
Two-boundary Temperley-Lieb algebras

[MNGB04] Fix $\delta, \delta_0, \delta_k \in \mathbb{C}$. The two-boundary Temperley-Lieb algebra $TL_k^{(2)}$ is a diagram algebra generated over $\mathbb{C}$ by diagrams $e_0 = \begin{tikzpicture}[baseline={([yshift=-.5ex]current bounding box.center)}]
\draw[thick, dashed, color=blue] (0,0) -- (1,0);
\draw[thick, dashed, color=blue] (0,0.5) -- (1,0.5);
\draw[thick, dashed, color=blue] (0,1) -- (1,1);
\fill (0,0) circle (2pt);
\fill (1,0) circle (2pt);
\fill (0,0.5) circle (2pt);
\fill (1,0.5) circle (2pt);
\fill (0,1) circle (2pt);
\fill (1,1) circle (2pt);
\end{tikzpicture}$, $e_k = \begin{tikzpicture}[baseline={([yshift=-.5ex]current bounding box.center)}]
\draw[thick, dashed, color=blue] (0,0) -- (1,0);
\draw[thick, dashed, color=blue] (0,0.5) -- (1,0.5);
\draw[thick, dashed, color=blue] (0,1) -- (1,1);
\fill (0,0) circle (2pt);
\fill (1,0) circle (2pt);
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\fill (0,0.5) circle (2pt);
\fill (1,0.5) circle (2pt);
\fill (0,1) circle (2pt);
\fill (1,1) circle (2pt);
\end{tikzpicture}$, and $e_i = \begin{tikzpicture}[baseline={([yshift=-.5ex]current bounding box.center)}]
\draw[thick, dashed, color=blue] (0,0) -- (1,0);
\draw[thick, dashed, color=blue] (0,0.5) -- (1,0.5);
\draw[thick, dashed, color=blue] (0,1) -- (1,1);
\fill (0,0) circle (2pt);
\fill (1,0) circle (2pt);
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\fill (1,0.5) circle (2pt);
\fill (0,1) circle (2pt);
\end{tikzpicture}$

for $i = 1, \ldots, k - 1$, with relations $e_ie_j = e_je_i$ for $|i - j| > 1$,

$e_ie_{i\pm 1}e_i = e_i$

for $1 \leq i \leq k - 1$,

$e_i^2 = \delta_i e_i$.

(Side loops are resolved with a 1 or a $\delta_i$ depending on whether there are an even or odd number of connections below their lowest point.)
Diagram multiplication:

\[ \delta \ast 1 \ast \delta^k \]

In short, \( \text{TL}(2)^k \) has basis given by non-crossing diagrams with

1. \( k \) connections to the top and to the bottom,
2. an even number of connections to the right and to the left, and
3. no edges with both ends on the left or both ends on the right.

However, \( 2 \ell \in \text{TL}(2)^k \) so unlike the classical T-L algebras, \( \text{TL}(2)^k \) is not finite dimensional!

Take quotient giving
Diagram multiplication:

In short, \( TL(2)_k \) has basis given by non-crossing diagrams with

1. \((1)\) \( k \) connections to the top and to the bottom,
2. \((2)\) an even number of connections to the right and to the left,
3. \((3)\) no edges with both ends on the left or both ends on the right.

However, \( 2^\ell \in TL(2)_k \)

So unlike the classical T-L algebras, \( TL(2)_k \) is not finite dimensional!

Take quotient giving \( z \)
Diagram multiplication:

In short, $\text{TL}(2)_k$ has basis given by non-crossing diagrams with

1. $(1)$ connections to the top and to the bottom,
2. $(2)$ an even number of connections to the right and to the left, and
3. $(3)$ no edges with both ends on the left or both ends on the right.

However, $2^{\ell} \in \text{TL}(2)_k$ So unlike the classical T-L algebras, $\text{TL}(2)_k$ is not finite dimensional!

Take quotient giving $z$
Diagram multiplication:

In short, \( TL(2)^k \) has basis given by non-crossing diagrams with

1. \( k \) connections to the top and to the bottom,
2. an even number of connections to the right and to the left, and
3. no edges with both ends on the left or both ends on the right.

However, \( 2^\ell \in TL(2)^k \)

So unlike the classical T-L algebras, \( TL(2)^k \) is not finite dimensional!

Take quotient giving \( = z \)
Diagram multiplication:

\[ \ast \delta \ast 1 \ast \delta_k \]

In short, \( T_L(2)_k \) has basis given by non-crossing diagrams with

1. \( k \) connections to the top and to the bottom,
2. an even number of connections to the right and to the left,
3. no edges with both ends on the left or both ends on the right.

However, \( 2^\ell \in T_L(2)_k \) so unlike the classical T-L algebras, \( T_L(2)_k \) is not finite dimensional!

Take quotient giving

\[ \ast \delta \ast \tau \ast \delta_k \]

Diagram multiplication:

\[ \begin{array}{c}
\text{Diagram multiplication:} \\
\end{array} \]

\[ \begin{array}{c}
\text{In short, } TL^{(2)}_k \text{ has basis given by non-crossing diagrams with} \\
(1) \text{ } k \text{ connections to the top and to the bottom,} \\
(2) \text{ an even number of connections to the right and to the left, and} \\
(3) \text{ no edges with both ends on the left or both ends on the right.}
\end{array} \]
Diagram multiplication:

\[
\begin{array}{c}
\text{Diagram multiplication:} \\
\end{array}
\]

In short, $TL_k^{(2)}$ has basis given by non-crossing diagrams with
(1) $k$ connections to the top and to the bottom,
(2) an even number of connections to the right and to the left, and
(3) no edges with both ends on the left or both ends on the right.

However,

\[
\begin{array}{c}
2\ell \\
\in TL_k^{(2)}
\end{array}
\]

So unlike the classical T-L algebras, $TL_k^{(2)}$ is not finite dimensional!
Diagram multiplication:

\[
\begin{array}{c}
\text{Diagram multiplication:} \\
\end{array}
\]

\[
= \begin{array}{c}
\text{Diagram} \\
\end{array} = \delta_k \ast 1 \ast \delta_k
\]

In short, $T L_{(2)}^{(2)}$ has basis given by non-crossing diagrams with

1. $k$ connections to the top and to the bottom,
2. an even number of connections to the right and to the left, and
3. no edges with both ends on the left or both ends on the right.

However, $2\ell \in T L_{(2)}^{(2)}$

So unlike the classical T-L algebras, $T L_{(2)}^{(2)}$ is not finite dimensional! Take quotient giving

\[
= z
\]
Representation theory of $TL_k^{(2)}$: action on diagrams

$d =$

\[ \begin{array}{c}
\text{Diagram}
\end{array} \]
Representation theory of $\mathcal{TL}_k^{(2)}$: action on diagrams
Representation theory of $TL_k^{(2)}$: action on diagrams

$d = \ldots$

$e_1d = \ldots = \delta_0$
Representation theory of $TL_{k}^{(2)}$: action on diagrams
Representation theory of $TL_k^{(2)}$: action on diagrams

\[ d = \]

\[ e_1d = \quad = \delta_0 \quad e_4d = \quad = \]
Representation theory of $TL_k^{(2)}$: action on diagrams

\[
\begin{align*}
    d &= \quad \\
    e_1d &= \quad = \delta_0 \\
    e_4d &= \quad = \\
    e_3e_4d &= \\
\end{align*}
\]
Representation theory of $\mathcal{TL}_k^{(2)}$: action on diagrams

$e_1d = d = \delta_0$

$e_4d = z$

$e_3e_4d = z$

$e_1d = d = \delta_0$

$e_4d = z$

$e_3e_4d = z$
Representation theory of $TL_k^{(2)}$: action on diagrams

\[ d = \]

\[ e_1 d = \delta_0 = e_4 d = \]

\[ e_3 e_4 = z \]
Representation theory of $\mathcal{T}L_{k}^{(2)}$: action on diagrams

\[ d = \]

\[ e_1 d = e_1 \]
\[ = \delta_0 \]
\[ = e_4 d = e_4 \]
\[ = \]

\[ e_3 e_4 d = e_3 e_4 \]
\[ = \]
\[ = z \]
Representation theory of $TL_k^{(2)}$: half diagrams

\[ d = \]

\[ e_1 d = e_4 d = \]

\[ e_3 e_4 d = z \]
Representation theory of $\mathcal{TL}_k^{(2)}$: half diagrams

$e_1 d = \delta_0$,

$e_4 d = \delta_0$,

$e_3 e_4 d = \delta_0$,

You can tell when to use $\delta_0$ or not by the parity of connections to the left/right walls.
Representation theory of $\mathcal{TL}^{(2)}_k$: half diagrams

\begin{equation*}
d = \begin{array}{c}
\end{array}
\end{equation*}

\begin{equation*}
e_1d = \begin{array}{c}
\end{array} = \delta_0 \begin{array}{c}
\end{array} \quad e_4d = \begin{array}{c}
\end{array} = \begin{array}{c}
\end{array}
\end{equation*}

\begin{equation*}
e_3e_4d = \begin{array}{c}
\end{array} = \begin{array}{c}
\end{array} = z \begin{array}{c}
\end{array}
\end{equation*}

You can tell when to use

\begin{equation*}
\begin{array}{c}
\end{array} = z
\end{equation*}

or not by the parity of connections to the left/right walls.
Representation theory of $TL_k^{(2)}$: half diagrams

\[ d = \]

\[ e_1d = \]  
\[ = \delta_0 \]  
\[ e_4d = \]  
\[ = \]

\[ e_3e_4d = \]  
\[ = \]  
\[ = z \]

You can tell when to use

\[ = z \]

or not by the parity of connections to the left/right walls.
Generic module:
(act by $e_i$, don’t make loops)
Generic module:
(act by $e_i$, don’t make loops)
Generic module:
(act by $e_i$, don’t make loops)
Red arrows indicate coef of $z$. 
Generic module:
(act by $e_i$, don’t make loops)
Red arrows indicate coef of $z$. 
Generic module:
(act by $e_i$, don’t make loops)
Red arrows indicate coef of $z$. 

\[ e_4 \]
\[ e_3 \]
\[ e_4 \]
\[ e_2 \]
Generic module:
(act by $e_i$, don’t make loops)
Red arrows indicate coef of $z$. 
**Generic module:**

(act by $e_i$, don’t make loops)

Red arrows indicate coef of $z$.

For what $z$ does this module split?
(1) The two-boundary (two-pole) braid group $B_k$ is generated by

$$T_k = \begin{array}{c}
\includegraphics[scale=0.5]{TK.pdf}
\end{array}, \quad T_0 = \begin{array}{c}
\includegraphics[scale=0.5]{T0.pdf}
\end{array} \text{ and } T_i = \begin{array}{c}
\includegraphics[scale=0.5]{Ti.pdf}
\end{array} \quad \text{for } 1 \leq i \leq k - 1,$$

subject to relations

$$\begin{array}{c}
\includegraphics[scale=0.5]{T0T1T2.pdf} \quad \begin{array}{c}
\includegraphics[scale=0.5]{Tk-2Tk-1Tk.pdf}
\end{array}
\end{array}.$$
(1) The two-boundary (two-pole) braid group $B_k$ is generated by

$$T_k = \begin{array}{c}
\begin{array}{c}
\text{\includegraphics{tikz/braid.png}}
\end{array}
\end{array}, \quad T_0 = \begin{array}{c}
\begin{array}{c}
\text{\includegraphics{tikz/braid.png}}
\end{array}
\end{array} \quad \text{and} \quad T_i = \begin{array}{c}
\begin{array}{c}
\text{\includegraphics{tikz/braid.png}}
\end{array}
\end{array} \quad \text{for } 1 \leq i \leq k - 1,$$

subject to relations

$$T_0 \begin{array}{c}
\begin{array}{c}
\text{\includegraphics{tikz/braid.png}}
\end{array}
\end{array} T_1 \begin{array}{c}
\begin{array}{c}
\text{\includegraphics{tikz/braid.png}}
\end{array}
\end{array} T_2 \begin{array}{c}
\begin{array}{c}
\text{\includegraphics{tikz/braid.png}}
\end{array}
\end{array} T_{k-2} \begin{array}{c}
\begin{array}{c}
\text{\includegraphics{tikz/braid.png}}
\end{array}
\end{array} T_{k-1} \begin{array}{c}
\begin{array}{c}
\text{\includegraphics{tikz/braid.png}}
\end{array}
\end{array} T_k \begin{array}{c}
\begin{array}{c}
\text{\includegraphics{tikz/braid.png}}
\end{array}
\end{array}.$$ 

(2) Fix constants $q_0, q_k, q \in \mathbb{C}$. The affine type C Hecke algebra $\mathcal{H}_k$ is the quotient of $\mathbb{C}B_k$ by the relations

$$(T_0 - q_0)(T_0 + q_0^{-1}) = 0, \quad (T_k - q_k)(T_k + q_k^{-1}) = 0$$

and

$$(T_i - q)(T_i + q^{-1}) = 0 \quad \text{for } i = 1, \ldots, k - 1.$$
(1) The two-boundary (two-pole) braid group $\mathcal{B}_k$ is generated by
\[ T_k = \begin{array}{c}
\begin{array}{c}
\bigotimes
\end{array}
\end{array}, \quad T_0 = \begin{array}{c}
\begin{array}{c}
\bigotimes
\end{array}
\end{array} \quad \text{and} \quad T_i = \begin{array}{c}
\begin{array}{c}
\bigotimes
\end{array}
\end{array} \quad \text{for} \quad 1 \leq i \leq k - 1,
\]
subject to relations \[ T_0 T_1 T_2 \cdots T_{k-2} T_{k-1} T_k. \]

(2) Fix constants $q_0, q_k, q = q_1 = q_2 = \cdots = q_{k-1} \in \mathbb{C}$. The affine type C Hecke algebra $\mathcal{H}_k$ is the quotient of $\mathbb{C}\mathcal{B}_k$ by the relations $(T_i - q_i^{1/2})(T_i + q_i^{-1/2}) = 0$.

(3) Set \[ e_0 = q_0 - T_0 \]
\[ e_k = q_k - T_k \]
\[ e_i = q - T_i \]
so that $e_j^2 = z_j e_j$ (for good $z_j$).
(1) The two-boundary (two-pole) braid group $B_k$ is generated by

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One advantage of using braids:
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\[
M \otimes V \otimes V \otimes V \otimes V \otimes V \otimes N
\]

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M \otimes V \otimes V \otimes V \otimes V \otimes V \otimes N
\]

Move both poles to the left

\[
M \otimes N \otimes V \otimes V \otimes V \otimes V \otimes V \otimes V \otimes V
\]

\[
M \otimes N \otimes V \otimes V \otimes V \otimes V \otimes V \otimes V \otimes V
\]

Jucys-Murphy elements:

\[
Z_i = i
\]

Pairwise commute

\[
Z(H_k) \text{ is (type-C) symmetric}
\]

Laurent polynomials in \(Z_i\)'s

\[
\text{Central characters indexed by } c \in \mathbb{C}
\]

\(k\) (modulo signed permutations)
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Jucys-Murphy elements:

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\[ Z_i(\lambda) \] is (type-C) symmetric Laurent polynomials in \( Z_i \)'s

Central characters indexed by \( c \in \mathbb{C} \) (modulo signed permutations)
One advantage of using braids:

Move both poles to the left

Jucys-Murphy elements:

$Z_i = \begin{array}{c}
\begin{array}{c}
\vdots
\end{array}
\end{array}$

Pairwise commute

Central characters indexed by $c \in C_k$ (modulo signed permutations)
One advantage of using braids:

Move both poles to the left:

Jucys-Murphy elements:

\[ Z_i = \]

- Pairwise commute
- \( Z(\mathcal{H}_k) \) is (type-C) symmetric
- Laurent polynomials in \( Z_i \)'s
One advantage of using braids:

- Pairwise commute
- $Z(\mathcal{H}_k)$ is (type-C) symmetric Laurent polynomials in $Z_i$’s
- Central characters indexed by $c \in \mathbb{C}^k$ (modulo signed permutations)

Jucys-Murphy elements:

$$Z_i = \begin{array}{ccccccc} & & & & & \cdots & \cdots \\ & & & & i & & i \\ & & & & & \cdots & \cdots \\ & & & & & \cdots & \cdots \\ \end{array}$$
Representation theory of $\mathcal{H}_k$

The representations of $\mathcal{H}_k$ are indexed by pairs $(c, J)$, where

- $c$ is a point in the fundamental chamber of the (finite) type C hyperplane system, and
- $J$ is a set of choices of positive/negative sides of other distinguished hyperplanes intersecting $c$.
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Example: $k = 2$
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Example: $k = 2$

$$c_2 = c_1 + 1$$

$$c_2 = c_1 - 1$$

$$c_2 = -c_1 + 1$$

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The $r_i$s depend on $\mathcal{H}_k$’s parameters $q_0$ and $q_k$: $r_1 = \log_q(q_0/q_k)$, $r_2 = \log_q(q_0q_k)$. 

![Diagram of representation theory of $\mathcal{H}_k$.](image-url)
The $r_i$s depend on $\mathcal{H}_k$'s parameters $q_0$ and $q_k$: $r_1 = \log_q(q_0/q_k)$, $r_2 = \log_q(q_0q_k)$. 

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Representation theory of $\mathcal{H}_k$

\[
\mathfrak{h}^{\alpha_2} \quad c_1 = r_1 \quad \mathfrak{h}^{\alpha_1}
\]

\[
c_2 = r_2
\]

\[
c_2 = r_1
\]

\[
c_2 = c_1 + 1
\]

\[
c_2 = -c_1 + 1
\]

The $r_i$s depend on $\mathcal{H}_k$'s parameters $q_0$ and $q_k$: $r_1 = \log_q(q_0/q_k)$, $r_2 = \log_q(q_0q_k)$. 

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The representation theory of $\mathcal{H}_k$.
A little more detail

- $J$ is determined by a set of positive roots (corresp. to hyperplanes).
- For “nice” characters, there is a bijection between alcoves and marked type-C generalized Young tableaux.
- “Intertwining operators” $\tau_i$ move between alcoves; dotted lines correspond to $\tau_i = 0$.

\[ J = \emptyset \quad \begin{array}{ccc} & -1 \ 2 & \end{array} \quad \begin{array}{ccc} & 1 \ 2 & \end{array} \quad \begin{array}{ccc} & -1 \ -2 & \end{array} \]

\[ J = \{ \varepsilon_2 - \varepsilon_1 \} \]

\[ J = \{ \varepsilon_2 \} \quad \begin{array}{ccc} & -2 \ -1 & \end{array} \quad \begin{array}{ccc} & 1 \ 2 & \end{array} \quad \begin{array}{ccc} & -2 \ -1 & \end{array} \]

\[ J = \{ \varepsilon, \varepsilon_2 - \varepsilon_1 \} \]
A little more detail

- $J$ is determined by a set of positive roots (corresp. to hyperplanes).
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\[
J = \emptyset \\
J = \{\varepsilon_2 - \varepsilon_1\} \\
J = \{\varepsilon_2\} \\
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Thm. (D.-Ram)

(1) Representations of $\mathcal{H}_k$ are indexed by pairs $(c, J)$. 
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(2) The representations of $\mathcal{H}_k$ that factor through the Temperley-Lieb quotient are as above.
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(1) Representations of $\mathcal{H}_k$ are indexed by pairs $(c, J)$.
(2) The representations of $\mathcal{H}_k$ that factor through the Temperley-Lieb quotient are as above.
See also…

Specific:


General:

https://zdaugherty.ccnysites.cuny.edu

Thanks!