Representations of the two-boundary Temperley-Lieb algebras

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work in progress additionally with
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September 4, 2019
Two-boundary Temperley-Lieb algebras

Fix $z, \delta_0, \delta_k \in \mathbb{C}$. The two-boundary Temperley-Lieb algebra $TL_k$ is a diagram algebra generated over $\mathbb{C}$ by diagrams

$$e_0 = \begin{array}{c}
\includegraphics[width=2cm]{eq0.png}
\end{array}, \quad e_k = \begin{array}{c}
\includegraphics[width=2cm]{eqk.png}
\end{array}, \quad \text{and} \quad e_i = \begin{array}{c}
\includegraphics[width=2cm]{eqi.png}
\end{array}$$

for $i = 1, \ldots, k - 1$
Two-boundary Temperley-Lieb algebras

Fix $z, \delta_0, \delta_k \in \mathbb{C}$. The two-boundary Temperley-Lieb algebra $TL_k$ is a diagram algebra generated over $\mathbb{C}$ by diagrams $e_0, e_k, e_i$ for $i = 1, \ldots, k - 1$, with relations $e_i e_j = e_j e_i$ for $|i - j| > 1$,

$e_i e_{i \pm 1} e_i = e_i$

for $1 \leq i \leq k - 1$,

$e_i^2 = \delta_i e_i$. 
Two-boundary Temperley-Lieb algebras

Fix $z, \delta_0, \delta_k \in \mathbb{C}$. The two-boundary Temperley-Lieb algebra $TL_k$ is a diagram algebra generated over $\mathbb{C}$ by diagrams

$$e_0 = \begin{array}{c}
\begin{array}{c}
\text{Diagram 1}
\end{array}
\end{array}, \quad e_k = \begin{array}{c}
\begin{array}{c}
\text{Diagram k}
\end{array}
\end{array}, \quad \text{and} \quad e_i = \begin{array}{c}
\begin{array}{c}
\text{Diagram i}
\end{array}
\end{array}$$

for $i = 1, \ldots, k - 1$, with relations $e_i e_j = e_j e_i$ for $|i - j| > 1$,

$$e_i e_{i \pm 1} e_i = e_i$$

for $1 \leq i \leq k - 1$,

$$e_i^2 = \delta_i e_i.$$
Two-boundary Temperley-Lieb algebras

Fix $z, \delta_0, \delta_k \in \mathbb{C}$. The two-boundary Temperley-Lieb algebra $TL_k$ is a diagram algebra generated over $\mathbb{C}$ by diagrams

$e_0 = \begin{array}{c}
\begin{array}{c}
1 \\
1
\end{array}
\end{array}$, $e_k = \begin{array}{c}
\begin{array}{c}
k \\
k
\end{array}
\end{array}$, and $e_i = \begin{array}{c}
\begin{array}{c}
i \\
i
\end{array}
\end{array}$

for $i = 1, \ldots, k - 1$, with relations $e_i e_j = e_j e_i$ for $|i - j| > 1$, $e_i e_{i\pm1} e_i = e_i$ for $1 \leq i \leq k - 1$, $e_i^2 = \delta_i e_i$.

or

or

or

or

or

or

or
Two-boundary Temperley-Lieb algebras

Mitra, Nienhuis, De Gier, Batchelor (2004), De Gier, Nichols (2009): Fix $z, \delta_0, \delta_k \in \mathbb{C}$. The two-boundary Temperley-Lieb algebra $T L_k$ is a diagram algebra generated over $\mathbb{C}$ by diagrams

\[ e_0 = \begin{tikzpicture} \draw[thick] (0,0) -- (1,0) -- (1,1) -- (0,1) -- cycle; \end{tikzpicture}, \quad e_k = \begin{tikzpicture} \draw[thick] (0,0) -- (1,0) -- (1,1) -- (0,1) -- cycle; \end{tikzpicture}, \quad \text{and} \quad e_i = \begin{tikzpicture} \draw[thick] (0,0) -- (1,0) -- (1,1) -- (0,1) -- cycle; \end{tikzpicture} \]

for $i = 1, \ldots, k - 1$, with relations $e_i e_j = e_j e_i$ for $|i - j| > 1$,

$e_i e_{i \pm 1} e_i = e_i$

for $1 \leq i \leq k - 1$,

$e_i^2 = \delta_i e_i$.

(Side loops are resolved with a 1 or a $\delta_i$ depending on whether there are an even or odd number of connections below their lowest point.)
Diagram multiplication:

In short, $\mathcal{T}_L^k$ has basis given by non-crossing diagrams with

1. $k$ connections to the top and to the bottom,
2. an even number of connections to the right and to the left, and
3. no edges with both ends on the left or both ends on the right.

However, $2\ell \in \mathcal{T}_L^k$ so unlike the classical T-L algebras, $\mathcal{T}_L^k$ is not finite dimensional!
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However, $2^{\ell} \in \mathcal{T}_L^k$

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Take quotient giving
Diagram multiplication:

In short, $\text{TL}_k$ has basis given by non-crossing diagrams with

1. $k$ connections to the top and to the bottom,
2. an even number of connections to the right and to the left,
3. no edges with both ends on the left or both ends on the right.

However, $2^{\ell} \in \text{TL}_k$ So unlike the classical T-L algebras, $\text{TL}_k$ is not finite dimensional!

Take quotient giving $z = \ldots$
Diagram multiplication:

\[ \text{In short, } \mathcal{L}_k \text{ has basis given by non-crossing diagrams with}
\]
\begin{enumerate}
\item \(k\) connections to the top and to the bottom,
\item an even number of connections to the right and to the left, and
\item no edges with both ends on the left or both ends on the right.
\end{enumerate}

However, \(2\ell \in \mathcal{L}_k\) so unlike the classical T-L algebras, \(\mathcal{L}_k\) is not finite dimensional!

Take quotient giving \(= z\).
Diagram multiplication:

In short, $\text{TL}_k$ has basis given by non-crossing diagrams with

1. $k$ connections to the top and to the bottom,
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3. no edges with both ends on the left or both ends on the right.

However, $\mathbb{Z}^{2\ell} \in \text{TL}_k$

So unlike the classical T-L algebras, $\text{TL}_k$ is not finite dimensional!

Take quotient giving $\ast \delta \ast 1 \ast \delta_k$
Diagram multiplication:

\[
\begin{array}{c}
\text{Diagram multiplication:} \\
\end{array}
\]

In short, \( TL_k \) has basis given by non-crossing diagrams with

1. \( k \) connections to the top and to the bottom,
2. an even number of connections to the right and to the left, and
3. no edges with both ends on the left or both ends on the right.

However, \( 2 \ell \in TL_k \)

So unlike the classical T-L algebras, \( TL_k \) is not finite dimensional!
Diagram multiplication:

\[
\begin{array}{c}
\text{Diagram multiplication:} \\
\end{array}
\]

\[
\begin{array}{c}
\text{In short, } TL_k \text{ has basis given by non-crossing diagrams with} \\
(1) \ k \text{ connections to the top and to the bottom,} \\
(2) \text{ an even number of connections to the right and to the left, and} \\
(3) \text{ no edges with both ends on the left or both ends on the right.} \\
\text{However,} \\
\end{array}
\]

\[
\begin{array}{c}
2\ell \\
\in TL_k
\end{array}
\]

\[
\begin{array}{c}
\text{So unlike the classical T-L algebras, } TL_k \text{ is not finite dimensional!}
\end{array}
\]
Diagram multiplication:

\[
\begin{array}{c}
\begin{tikzpicture}
\end{tikzpicture}
\end{array}
\quad = \quad
\begin{array}{c}
\begin{tikzpicture}
\end{tikzpicture}
\end{array}
\quad \ast \delta \ast 1 \ast \delta_k
\]

In short, $T_Lk$ has basis given by non-crossing diagrams with

1. $k$ connections to the top and to the bottom,
2. an even number of connections to the right and to the left, and
3. no edges with both ends on the left or both ends on the right.

However,

\[
\begin{array}{c}
\begin{tikzpicture}
\end{tikzpicture}
\end{array}
\quad \in \quad T_Lk
\]

So unlike the classical T-L algebras, $T_Lk$ is not finite dimensional!

Take quotient giving

\[
\begin{array}{c}
\begin{tikzpicture}
\end{tikzpicture}
\end{array}
\quad = \quad z
\]
Representation theory of $\mathcal{T}L_k$: action on diagrams

d = \begin{array}{c}
\end{array}
Representation theory of $TL_k$: action on diagrams

$e_1d = \quad d =$
Representation theory of $\mathcal{TL}_k$: action on diagrams

\[ d = \]

\[ e_1 d = \]

\[ = \delta_0 \]
Representation theory of $T L_k$: action on diagrams

\[
d = \begin{array}{c}
\end{array}
\]

\[
e_1d = \begin{array}{c}
\end{array} = \delta_0 = \begin{array}{c}
\end{array}
\]

\[
e_4d = \begin{array}{c}
\end{array}
\]
Representation theory of $TL_k$: action on diagrams

$$d = \text{Diagram}$$

$$e_1d = \text{Diagram} = \delta_0$$

$$e_4d = \text{Diagram}$$
Representation theory of $TL_k$: action on diagrams

\[ d = \]

\[ e_1d = \quad = \delta_0 \quad e_4d = \quad = \]

\[ e_3e_4d = \]
Representation theory of $\mathcal{T}L_k$: action on diagrams

\[ e_1d = \delta_0 \]

\[ e_4d = \]

\[ e_3e_4d = \]
Representation theory of $TL_k$: action on diagrams

\[ d = \]

\[ e_1d = \]

\[ = \delta_0 \]

\[ e_4d = \]

\[ = \]

\[ e_3e_4d = \]

\[ = \]

\[ = z \]
Representation theory of $\mathcal{TL}_k$: action on diagrams

$$d = \begin{array}{c}
\end{array}$$

$$e_1d = \begin{array}{c}
\end{array} = \delta_0$$

$$e_4d = \begin{array}{c}
\end{array}$$

$$e_3e_4d = \begin{array}{c}
\end{array} = \begin{array}{c}
\end{array} = z$$
Representation theory of $T L_k$: half diagrams

\[ d = \]

\[ e_1 d = \]

\[ e_4 d = \]

\[ e_3 e_4 d = \]

You can tell when to use $=\delta_0$ or not by the parity of connections to the left/right walls.
Representation theory of $\mathcal{T}L_k$: half diagrams

d =

$e_1d = \delta_0 = e_4d = $

$e_3e_4d = $

You can tell when to use

or not by the parity of connections to the left/right walls.
Representation theory of $\mathcal{TL}_k$: half diagrams

$e_1d = \delta_0 = e_4d = \delta_0$

$e_3e_4d = z$

You can tell when to use $z$ or not by the parity of connections to the left/right walls.
Standard module:
(act by $e_i$, don’t make loops)
Standard module:
(act by $e_i$, don’t make loops)
Standard module:
(act by $e_i$, don’t make loops)
Red arrows indicate coef of $z$. 
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Red arrows indicate coef of $z$.

For what $z$ does this module split?
Actions on tensor space

Two-boundary Temperley-Lieb diagrams have a natural action on special tensor products of $U_q\mathfrak{sl}_2$-modules...
Actions on tensor space

Two-boundary Temperley-Lieb diagrams have a natural action on special tensor products of $U_q\mathfrak{sl}_2$-modules.

Let $V = L(\Box) = \mathbb{C}^2$, $M = L(a)$, $N = L(b)$ be highest-weight $U_q\mathfrak{sl}_2$-modules. Then $TL_k$ acts on

$$M \otimes V \otimes V \otimes V \otimes V \otimes V \otimes N$$
Two-boundary Temperley-Lieb diagrams have a natural action on special tensor products of $U_q \mathfrak{sl}_2$-modules.

Let $V = L(\square) = \mathbb{C}^2$, $M = L(a)$, $N = L(b)$ be highest-weight $U_q \mathfrak{sl}_2$-modules. Then $TL_k$ acts on

\[
M \otimes V \otimes V \otimes V \otimes V \otimes V \otimes V \otimes N
\]

via factor permutation and projection operators (the parameters depend on $q, a$, and $b$). Further, this action centralizes the action of $U_q \mathfrak{sl}_2$.
Schur-Weyl Duality

$\text{GL}_n(\mathbb{C})$ acts on $\mathbb{C}^n \otimes \mathbb{C}^n \otimes \cdots \otimes \mathbb{C}^n = (\mathbb{C}^n)^\otimes k$ diagonally.

$g \cdot (v_1 \otimes v_2 \otimes \cdots \otimes v_k) = gv_1 \otimes gv_2 \otimes \cdots \otimes gv_k.$
Schur-Weyl Duality

\( \text{GL}_n(\mathbb{C}) \) acts on \( \mathbb{C}^n \otimes \mathbb{C}^n \otimes \cdots \otimes \mathbb{C}^n = (\mathbb{C}^n)^\otimes k \) diagonally.

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\( S_k \) also acts on \( (\mathbb{C}^n)^\otimes k \) by place permutation.
Schur-Weyl Duality

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$S_k$ also acts on $(\mathbb{C}^n)^\otimes k$ by place permutation.

These actions commute!
Schur-Weyl Duality

\[
\text{End}_{\text{GL}_n} \left( (\mathbb{C}^n \otimes k) \right) = \pi(\mathbb{C} S_k) \quad \text{and} \quad \text{End}_{S_k} \left( (\mathbb{C}^n \otimes k) \right) = \rho(\mathbb{C} \text{GL}_n).
\]

(End of all linear maps that commute with \(\text{GL}_n\))

(img of \(S_k\) action)

(img of \(\text{GL}_n\) action)
Schur-Weyl Duality

\[
\text{End}_{\text{GL}_n} \left( (\mathbb{C}^n) \otimes_k \right) = \pi(\mathbb{C}S_k) \quad \text{and} \quad \text{End}_{S_k} \left( (\mathbb{C}^n) \otimes_k \right) = \rho(\mathbb{C}\text{GL}_n) .
\]

(all linear maps that commute with \( \text{GL}_n \))

(img of \( S_k \) action)

(img of \( \text{GL}_n \) action)

Powerful consequence: a duality between representations

The double-centralizer relationship produces

\[
(\mathbb{C}^n) \otimes_k \cong \bigoplus_{\lambda \vdash k} G^\lambda \otimes S^\lambda \quad \text{as a } \text{GL}_n-S_k \text{ bimodule,}
\]

where

\[
G^\lambda \quad \text{are distinct irreducible } \text{GL}_n\text{-modules}
\]

\[
S^\lambda \quad \text{are distinct irreducible } S_k\text{-modules}
\]
More centralizer algebras

Brauer (1937)
Orthogonal and symplectic groups
(and Lie algebras) acting on
\((\mathbb{C}^n)^\otimes k\) diagonally centralize
the **Brauer algebra**:

\[
\delta_{b,c} \sum_{i=1}^{n} v_i \otimes v_i \otimes v_a \otimes v_e \otimes v_d
\]

with \(\bigcirc = n\)

(Diagrams encode maps \(V^\otimes k \to V^\otimes k\) that commute with the action of some classical algebra.)
More centralizer algebras

Brauer (1937)
Orthogonal and symplectic groups
(and Lie algebras) acting on
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the Brauer algebra:

\[
\delta_{b,c} \sum_{i=1}^{n} v_i \otimes v_i \otimes v_a \otimes v_e \otimes v_d
\]
with \(\bigcirc = n\)

\[
\delta_{c,d} \sum_{i=1}^{2} v_a \otimes v_i \otimes v_i \otimes v_b \otimes v_e
\]
with \(\bigcirc = 2\)

Temperley-Lieb (1971)
\(\text{GL}_2\) and \(\text{SL}_2\) (and \(\mathfrak{gl}_2\) and \(\mathfrak{sl}_2\)) acting on \((\mathbb{C}^2)^k\) diagonally centralize
the Temperley-Lieb algebra:

(Diagrams encode maps \(V^\otimes k \to V^\otimes k\) that commute with the action of some classical algebra.)
Quantum groups and braids

Fix $q \in \mathbb{C}$, and let $\mathcal{U} = \mathcal{U}_q \mathfrak{g}$ be the Drinfeld-Jimbo quantum group associated to Lie algebra $\mathfrak{g}$. 

$\mathcal{U} \otimes \mathcal{U}$ has an invertible element $R = \sum R_{12} \otimes R_{21}$ that yields a map $\tilde{R} : V \otimes W \rightarrow W \otimes V$ that (1) satisfies braid relations, and (2) commutes with the $\mathcal{U}$-action on $V \otimes W$ for any $\mathcal{U}$-modules $V$ and $W$. 

The braid group shares a commuting action with $\mathcal{U}$ on $V \otimes k$:

Around the pole: $M \otimes V M \otimes V \rightarrow \tilde{R} M V \tilde{R} V M$.
Quantum groups and braids

Fix $q \in \mathbb{C}$, and let $\mathcal{U} = \mathcal{U}_q \mathfrak{g}$ be the Drinfeld-Jimbo quantum group associated to Lie algebra $\mathfrak{g}$.

$\mathcal{U} \otimes \mathcal{U}$ has an invertible element $\mathcal{R} = \sum_{\mathcal{R}} R_1 \otimes R_2$ that yields a map

$$\tilde{\mathcal{R}}_{VW} : V \otimes W \longrightarrow W \otimes V$$

that

1. satisfies braid relations, and
2. commutes with the $\mathcal{U}$-action on $V \otimes W$ for any $\mathcal{U}$-modules $V$ and $W$. 

Quantum groups and braids

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Quantum groups and braids

Fix \( q \in \mathbb{C} \), and let \( \mathcal{U} = \mathcal{U}_q \mathfrak{g} \) be the Drinfeld-Jimbo quantum group associated to Lie algebra \( \mathfrak{g} \).

\( \mathcal{U} \otimes \mathcal{U} \) has an invertible element \( R = \sum R_1 \otimes R_2 \) that yields a map

\[
\tilde{R}_{VW} : V \otimes W \rightarrow W \otimes V
\]

that

1. satisfies braid relations, and
2. commutes with the \( \mathcal{U} \)-action on \( V \otimes W \)

for any \( \mathcal{U} \)-modules \( V \) and \( W \).

The one-pole/affine braid group shares a commuting action with \( \mathcal{U} \) on \( M \otimes V \otimes^k \):

Around the pole:

\[
M \otimes V = \tilde{R}_{MV} \tilde{R}_{VM}
\]
Quantum groups and braids

Fix $q \in \mathbb{C}$, and let $\mathcal{U} = \mathcal{U}_q \mathfrak{g}$ be the Drinfeld-Jimbo quantum group associated to Lie algebra $\mathfrak{g}$.

$\mathcal{U} \otimes \mathcal{U}$ has an invertible element $\mathcal{R} = \sum \mathcal{R}_{1} \otimes \mathcal{R}_{2}$ that yields a map

$$\tilde{\mathcal{R}}_{VW} : V \otimes W \to W \otimes V$$

that
(1) satisfies braid relations, and
(2) commutes with the $\mathcal{U}$-action on $V \otimes W$

for any $\mathcal{U}$-modules $V$ and $W$.

The two-pole braid group shares a commuting action with $\mathcal{U}$ on $M \otimes V^\otimes k \otimes N$:

Around the pole:

$$= \tilde{\mathcal{R}}_{MV} \tilde{\mathcal{R}}_{VM}$$
The two-boundary (two-pole) braid group $B_k$ is generated by

\[ T_k = \begin{array}{c} \text{Diagram} \end{array}, \quad T_0 = \begin{array}{c} \text{Diagram} \end{array} \quad \text{and} \quad T_i = \begin{array}{c} \text{Diagram} \end{array} \quad \text{for } 1 \leq i \leq k - 1, \]
The two-boundary (two-pole) braid group $\mathcal{B}_k$ is generated by

$$T_k = \begin{array}{c}
\text{\includegraphics[width=0.2\textwidth]{braid1.pdf}}
\end{array}, \quad T_0 = \begin{array}{c}
\text{\includegraphics[width=0.2\textwidth]{braid2.pdf}}
\end{array} \quad \text{and} \quad T_i = \begin{array}{c}
\text{\includegraphics[width=0.2\textwidth]{braid3.pdf}}
\end{array} \quad \text{for } 1 \leq i \leq k - 1,$$

subject to relations

$$T_i T_{i+1} T_i = \begin{array}{c}
\text{\includegraphics[width=0.5\textwidth]{braid4.pdf}}
\end{array} = \begin{array}{c}
\text{\includegraphics[width=0.5\textwidth]{braid5.pdf}}
\end{array} = T_{i+1} T_i T_{i+1},$$
The two-boundary (two-pole) braid group $B_k$ is generated by

$$T_k = \begin{array}{c}
\begin{array}{c}
\text{\textbullet} \\
\text{\textbullet}
\end{array}
\end{array}, \quad T_0 = \begin{array}{c}
\begin{array}{c}
\text{\textbullet} \\
\text{\textbullet}
\end{array}
\end{array} \quad \text{and} \quad T_i = \begin{array}{c}
\begin{array}{c}
\text{\textbullet} \\
\text{\textbullet}
\end{array}
\end{array} \quad \text{for } 1 \leq i \leq k - 1,$$

subject to relations

$$T_i T_{i+1} T_i = \begin{array}{c}
\begin{array}{c}
\text{\textbullet} \\
\text{\textbullet}
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\text{\textbullet} \\
\text{\textbullet}
\end{array}
\end{array} = T_{i+1} T_i T_{i+1},$$

$$T_1 T_0 T_1 T_0 = \begin{array}{c}
\begin{array}{c}
\text{\textbullet} \\
\text{\textbullet}
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\text{\textbullet} \\
\text{\textbullet}
\end{array}
\end{array} = T_0 T_1 T_0 T_1,$$
The two-boundary (two-pole) braid group $B_k$ is generated by

$$T_k = \begin{array}{c} \begin{array}{c} \text{Diagram of a braid}\end{array} \end{array}, \quad T_0 = \begin{array}{c} \begin{array}{c} \text{Diagram of a braid}\end{array} \end{array} \quad \text{and} \quad T_i = \begin{array}{c} \begin{array}{c} \text{Diagram of a braid}\end{array} \end{array} \quad \text{for } 1 \leq i \leq k - 1,$$

subject to relations

$$T_iT_{i+1}T_i = \begin{array}{c} \begin{array}{c} \text{Diagram of braid relations}\end{array} \end{array} = \begin{array}{c} \begin{array}{c} \text{Diagram of braid relations}\end{array} \end{array} = T_{i+1}T_iT_{i+1},$$

$$T_1T_0T_1T_0 = \begin{array}{c} \begin{array}{c} \text{Diagram of braid relations}\end{array} \end{array} = \begin{array}{c} \begin{array}{c} \text{Diagram of braid relations}\end{array} \end{array} = T_0T_1T_0T_1,$$

and, similarly, $T_{k-1}T_kT_{k-1}T_k = T_kT_{k-1}T_kT_{k-1}$.
The two-boundary (two-pole) braid group $\mathcal{B}_k$ is generated by

$$T_k = \begin{tikzpicture}[baseline=-.5ex]
\draw (-.2,0) -- (-.2,2);
\draw (.2,0) -- (.2,2);
\draw (0,1.5) -- (0,2);
\draw (0,1) -- (0,0);
\draw (0,1) -- (0,2);
\end{tikzpicture}, \quad T_0 = \begin{tikzpicture}[baseline=-.5ex]
\draw (-.5,0) -- (-.5,2);
\draw (.5,0) -- (.5,2);
\draw (0,1.5) -- (0,2);
\draw (0,1) -- (0,0);
\draw (0,1) -- (0,2);
\end{tikzpicture} \quad \text{and} \quad T_i = \begin{tikzpicture}[baseline=-.5ex]
\draw (-.2,0) -- (-.2,2);
\draw (.2,0) -- (.2,2);
\draw (0,1.5) -- (0,2);
\draw (0,1) -- (0,0);
\draw (0,1) -- (0,2);
\end{tikzpicture} \quad \text{for } 1 \leq i \leq k - 1,$$

subject to relations

$$T_0 T_1 T_2 \cdots T_{k-2} T_{k-1} T_k$$

i.e.

$$T_i T_{i+1} T_i = \begin{tikzpicture}[baseline=-.5ex]
\draw (-.2,0) -- (-.2,2);
\draw (.2,0) -- (.2,2);
\draw (0,1.5) -- (0,2);
\draw (0,1) -- (0,0);
\draw (0,1) -- (0,2);
\end{tikzpicture} = \begin{tikzpicture}[baseline=-.5ex]
\draw (-.2,0) -- (-.2,2);
\draw (.2,0) -- (.2,2);
\draw (0,1.5) -- (0,2);
\draw (0,1) -- (0,0);
\draw (0,1) -- (0,2);
\end{tikzpicture} = \begin{tikzpicture}[baseline=-.5ex]
\draw (-.2,0) -- (-.2,2);
\draw (.2,0) -- (.2,2);
\draw (0,1.5) -- (0,2);
\draw (0,1) -- (0,0);
\draw (0,1) -- (0,2);
\end{tikzpicture},$$

$$T_1 T_0 T_1 T_0 = \begin{tikzpicture}[baseline=-.5ex]
\draw (-.2,0) -- (-.2,2);
\draw (.2,0) -- (.2,2);
\draw (0,1.5) -- (0,2);
\draw (0,1) -- (0,0);
\draw (0,1) -- (0,2);
\end{tikzpicture} = \begin{tikzpicture}[baseline=-.5ex]
\draw (-.2,0) -- (-.2,2);
\draw (.2,0) -- (.2,2);
\draw (0,1.5) -- (0,2);
\draw (0,1) -- (0,0);
\draw (0,1) -- (0,2);
\end{tikzpicture} = \begin{tikzpicture}[baseline=-.5ex]
\draw (-.2,0) -- (-.2,2);
\draw (.2,0) -- (.2,2);
\draw (0,1.5) -- (0,2);
\draw (0,1) -- (0,0);
\draw (0,1) -- (0,2);
\end{tikzpicture},$$

and, similarly, $T_{k-1} T_k T_{k-1} T_k = T_k T_{k-1} T_k T_{k-1}$. 
(1) The two-boundary (two-pole) braid group $B_k$ is generated by $T_k$, $T_0$, and $T_i$ for $1 \leq i \leq k - 1$, subject to relations $T_0 T_1 T_2 \cdots T_{k-2} T_{k-1} T_k$.
(1) The two-boundary (two-pole) braid group $B_k$ is generated by
\[
T_k = \begin{array}{c}
\text{------} \\
\text{-----} \\
\text{-----} \\
\text{-----}
\end{array}, \quad T_0 = \begin{array}{c}
\text{-----} \\
\text{-----} \\
\text{-----} \\
\text{-----}
\end{array} \quad \text{and} \quad T_i = \begin{array}{c}
\text{-----} \\
\text{-----} \\
\text{-----} \\
\text{-----}
\end{array} \quad \text{for } 1 \leq i \leq k - 1,
\]
subject to relations
\[
\begin{array}{c}
T_0 T_1 = T_1 T_0, \\
T_2 T_1 = T_1 T_2, \\
\end{array} \quad T_k T_{k-1} = T_{k-1} T_k.
\]

(2) Fix constants $t_0, t_k, t \in \mathbb{C}$. The affine type C Hecke algebra $H_k$ is the quotient of $\mathbb{C}B_k$ by the relations
\[
(T_0 - t_0^{1/2})(T_0 + t_0^{-1/2}) = 0, \quad (T_k - t_k^{1/2})(T_k + t_k^{-1/2}) = 0
\]
and
\[
(T_i - t^{1/2})(T_i + t^{-1/2}) = 0 \quad \text{for } i = 1, \ldots, k - 1.
\]
(1) The two-boundary (two-pole) braid group $B_k$ is generated by

\[
T_k = \begin{array}{c}
\includegraphics{two-boundary_braid.png}
\end{array}, \quad T_0 = \begin{array}{c}
\includegraphics{two-boundary_braid.png}
\end{array} \quad \text{and} \quad T_i = \begin{array}{c}
\includegraphics{two-boundary_braid.png}
\end{array} \quad \text{for } 1 \leq i \leq k - 1,
\]

subject to relations

\[
\begin{array}{c}
\includegraphics{two-boundary_relations.png}
\end{array}.
\]

(2) Fix constants $t_0, t_k, t = t_1 = t_2 = \cdots = t_{k-1} \in \mathbb{C}$. The affine type C Hecke algebra $\mathcal{H}_k$ is the quotient of $\mathbb{C}B_k$ by the relations

\[(T_i - t_i^{1/2})(T_i + t_i^{-1/2}) = 0.\]
(1) The two-boundary (two-pole) braid group $B_k$ is generated by

$$
T_k = \begin{array}{c}
\circ \\
\circ
\end{array}, \quad T_0 = \begin{array}{c}
\circ \\
\circ
\end{array} \quad \text{and} \quad T_i = \begin{array}{c}
\circ \\
\circ
\end{array} \quad \text{for} \quad 1 \leq i \leq k - 1,
$$

subject to relations

$$
\begin{array}{cccccccc}
\circ & = & \circ & \circ & \circ & \circ & \circ & \circ \\
\circ & = & \circ & \circ & \circ & \circ & \circ & \circ
\end{array}.
$$

(2) Fix constants $t_0, t_k, t = t_1 = t_2 = \cdots = t_{k - 1} \in \mathbb{C}$. The affine type C Hecke algebra $\mathcal{H}_k$ is the quotient of $\mathbb{C}B_k$ by the relations $(T_i - t_i^{1/2})(T_i + t_i^{-1/2}) = 0$.

(3) Set

$$
\begin{array}{c}
\circ \\
\circ
\end{array} = t_0^{1/2} \begin{array}{c}
\circ \\
\circ
\end{array} - \begin{array}{c}
\circ \\
\circ
\end{array} \quad (e_0 = t_0^{1/2} - T_0)
$$

$$
\begin{array}{c}
\circ \\
\circ
\end{array} = t_k^{1/2} \begin{array}{c}
\circ \\
\circ
\end{array} - \begin{array}{c}
\circ \\
\circ
\end{array} \quad (e_k = t_k^{1/2} - T_k)
$$

$$
\begin{array}{c}
\circ \\
\circ
\end{array} = t_i^{1/2} \begin{array}{c}
\circ \\
\circ
\end{array} - \begin{array}{c}
\circ \\
\circ
\end{array} \quad (e_i = t_i^{1/2} - T_i)
$$

so that $e_j^2 = z_j e_j$ (for good $z_j$).
(1) The two-boundary (two-pole) braid group $B_k$ is generated by

$$T_k = \begin{array}{cc}
\bullet & \\
\bullet & \end{array}, \quad T_0 = \begin{array}{cc}
\bullet & \\
\bullet & \end{array} \quad \text{and} \quad T_i = \begin{array}{cc}
\bullet & \\
\bullet & \end{array} \quad \text{for } 1 \leq i \leq k - 1,$$

subject to relations

$$T_0 T_1 T_2 \cdots T_{k-2} T_{k-1} T_k.$$ 

(2) Fix constants $t_0, t_k, t = t_1 = t_2 = \cdots = t_{k-1} \in \mathbb{C}$. The affine type C Hecke algebra $\mathcal{H}_k$ is the quotient of $\mathbb{C}B_k$ by the relations $(T_i - t_i^{1/2})(T_i + t_i^{-1/2}) = 0$.

(3) Set

$$e_0 = t_0^{1/2} - T_0 \quad (e_0 = t_0^{1/2} - T_0)$$

$$e_k = t_k^{1/2} - T_k \quad (e_k = t_k^{1/2} - T_k)$$

$$e_i = t_i^{1/2} - T_i \quad (e_i = t_i^{1/2} - T_i)$$

so that $e_j^2 = z_j e_j$ (for good $z_j$).

The two-boundary Temperley-Lieb algebra is the quotient of $\mathcal{H}_k$ by the relations $e_i e_{i \pm 1} e_i = e_i$ for $i = 1, \ldots, k - 1$. 
(1) The two-boundary (two-pole) braid group $B_k$ is generated by

\[ T_k = \begin{array}{c}
\text{\includegraphics[width=0.1\textwidth]{braid1.png}} \\
\text{\includegraphics[width=0.1\textwidth]{braid2.png}} \end{array} \text{, } T_0 = \begin{array}{c}
\text{\includegraphics[width=0.1\textwidth]{braid3.png}} \\
\text{\includegraphics[width=0.1\textwidth]{braid4.png}} \end{array} \text{ and } T_i = \begin{array}{c}
\text{\includegraphics[width=0.1\textwidth]{braid5.png}} \\
\text{\includegraphics[width=0.1\textwidth]{braid6.png}} \end{array} \text{ for } 1 \leq i \leq k - 1. \]

(2) Fix constants $t_0, t_k, t = t_1 = t_2 = \cdots = t_{k-1} \in \mathbb{C}$. The affine type C Hecke algebra $H_k$ is the quotient of $\mathbb{C}B_k$ by the relations $(T_i - t_i^{1/2})(T_i + t_i^{-1/2}) = 0$.

(3) Set

\[ \text{\includegraphics[width=0.1\textwidth]{hecke1.png}} = t_0^{1/2} \begin{array}{c}
\text{\includegraphics[width=0.1\textwidth]{braid1.png}} \\
\text{\includegraphics[width=0.1\textwidth]{braid2.png}} \end{array} - \begin{array}{c}
\text{\includegraphics[width=0.1\textwidth]{braid3.png}} \\
\text{\includegraphics[width=0.1\textwidth]{braid4.png}} \end{array} \text{, } \text{\includegraphics[width=0.1\textwidth]{hecke2.png}} = t_k^{1/2} \begin{array}{c}
\text{\includegraphics[width=0.1\textwidth]{braid1.png}} \\
\text{\includegraphics[width=0.1\textwidth]{braid2.png}} \end{array} - \begin{array}{c}
\text{\includegraphics[width=0.1\textwidth]{braid3.png}} \\
\text{\includegraphics[width=0.1\textwidth]{braid4.png}} \end{array} \text{ and } \text{\includegraphics[width=0.1\textwidth]{hecke3.png}} = t^{1/2} \begin{array}{c}
\text{\includegraphics[width=0.1\textwidth]{braid1.png}} \\
\text{\includegraphics[width=0.1\textwidth]{braid2.png}} \end{array} - \begin{array}{c}
\text{\includegraphics[width=0.1\textwidth]{braid3.png}} \\
\text{\includegraphics[width=0.1\textwidth]{braid4.png}} \end{array} \]

so that $e_j^2 = z_j e_j$. The two-boundary Temperley-Lieb algebra is the quotient of $H_k$ by the relations $e_i e_{i\pm 1} e_i = e_i$ for $i = 1, \ldots, k - 1$. 
(1) The two-boundary (two-pole) braid group \( \mathcal{B}_k \) is generated by

\[
T_k = \begin{array}{c}
\begin{array}{c}
\includegraphics{braid1.png}
\end{array}
\end{array}, \quad T_0 = \begin{array}{c}
\begin{array}{c}
\includegraphics{braid2.png}
\end{array}
\end{array} \quad \text{and} \quad T_i = \begin{array}{c}
\begin{array}{c}
\includegraphics{braid3.png}
\end{array}
\end{array} \quad \text{for } 1 \leq i \leq k - 1.
\]

(2) Fix constants \( t_0, t_k, t = t_1 = t_2 = \cdots = t_{k-1} \in \mathbb{C} \).
The affine type C Hecke algebra \( \mathcal{H}_k \) is the quotient of \( \mathbb{C}\mathcal{B}_k \) by the relations \((T_i - t_i^{1/2})(T_i + t_i^{-1/2}) = 0\).

(3) Set

\[
\begin{array}{c}
\begin{array}{c}
\includegraphics{hecke1.png}
\end{array}
\end{array} = t_0^{1/2} \begin{array}{c}
\begin{array}{c}
\includegraphics{braid4.png}
\end{array}
\end{array} - \begin{array}{c}
\begin{array}{c}
\includegraphics{braid5.png}
\end{array}
\end{array}, \quad \begin{array}{c}
\begin{array}{c}
\includegraphics{hecke2.png}
\end{array}
\end{array} = t_k^{1/2} \begin{array}{c}
\begin{array}{c}
\includegraphics{braid6.png}
\end{array}
\end{array} - \begin{array}{c}
\begin{array}{c}
\includegraphics{braid7.png}
\end{array}
\end{array} \quad \text{and} \quad \begin{array}{c}
\begin{array}{c}
\includegraphics{hecke3.png}
\end{array}
\end{array} = t^{1/2} \begin{array}{c}
\begin{array}{c}
\includegraphics{braid8.png}
\end{array}
\end{array} - \begin{array}{c}
\begin{array}{c}
\includegraphics{braid9.png}
\end{array}
\end{array}
\]

so that \( e_j^2 = z_j e_j \). The two-boundary Temperley-Lieb algebra is the quotient of \( \mathcal{H}_k \) by the relations \( e_i e_{i+1} e_i = e_i \) for \( i = 1, \ldots, k - 1 \).
Move both poles to the left.

Jucys-Murphy elements: $Z_i = i$ pairwise commute. $Z_i(H_k)$ is (type-C) symmetric Laurent polynomials in $Z_i$'s. Central characters indexed by $c \in \mathbb{C}$, $k$ (modulo signed permutations).
Move both poles to the left

\[ Z_i = i \]

Pairwise commute
\[ Z(H_k) \]

is (type-C) symmetric

Laurent polynomials in \( Z_i \)’s

Central characters indexed by \( c \in \mathbb{C} \)

\( k \) (modulo signed permutations)
Jucys-Murphy elements:

\[ Z_i = \]

Move both poles to the left
Move both poles
to the left

Jucys-Murphy elements:

\[ Z_i = \]

- Pairwise commute
Move both poles to the left

Jucys-Murphy elements:

\[ Z_i = \]

- Pairwise commute
- \( Z(\mathcal{H}_k) \) is (type-C) symmetric
- Laurent polynomials in \( Z_i \)'s
Move both poles to the left

Jucys-Murphy elements:

$$Z_i = \begin{array}{c}
\end{array}$$

- Pairwise commute
- $Z(\mathcal{H}_k)$ is (type-C) symmetric
- Laurent polynomials in $Z_i$'s
- Central characters indexed by $c \in \mathbb{C}^k$ (modulo signed permutations)
Representation theory of $\mathcal{H}_k$

The representations of $\mathcal{H}_k$ are indexed by pairs $(c, J)$, where

$\begin{align*}
\text{c} & \text{ is a point in the fundamental chamber of} \\
\text{the (finite) type C hyperplane system, and} \\
\text{J} & \text{ is a set of choices of positive/negative sides of} \\
\text{other distinguished hyperplanes intersecting c}
\end{align*}$
Representation theory of $H_k$

The representations of $H_k$ are indexed by pairs $(c, J)$, where

- $c$ is a point in the fundamental chamber of the (finite) type C hyperplane system, and
- $J$ is a set of choices of positive/negative sides of other distinguished hyperplanes intersecting $c$

Example: $k = 2$

\[
\begin{align*}
\mathfrak{h}^{\alpha_1 + \alpha_2} & \quad \mathfrak{h}^{\alpha_2} & \quad \mathfrak{h}^{\alpha_1} \\
\mathfrak{h}^{\alpha_2} & \quad \mathfrak{h}^{\alpha_1} & \quad \mathfrak{h}^{\alpha_2 + 2\alpha_1}
\end{align*}
\]
Representation theory of $\mathcal{H}_k$

The representations of $\mathcal{H}_k$ are indexed by pairs $(c, J)$, where

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Example: $k = 2$

The $r_i$s depend on $\mathcal{H}_k$'s parameters $t_0$ and $t_k$: $r_1 = \log_t (t_0/t_k)$, $r_2 = \log_t (t_0 t_k)$
The $r_i$s depend on $\mathcal{H}_k$’s parameters $t_0$ and $t_k$: $r_1 = \log_t(t_0/t_k), \quad r_2 = \log_t(t_0 t_k)$
Representation theory of $\mathcal{H}_k$

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Representation theory of $\mathcal{H}_k$

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The $r_i$s depend on $\mathcal{H}_k$'s parameters $t_0$ and $t_k$: $r_1 = \log_t(t_0/t_k)$, $r_2 = \log_t(t_0t_k)$
A little more detail

- $J$ is determined by a set of positive roots (corresp. to hyperplanes).
- For “nice” characters, there is a bijection between alcoves and marked type-C generalized Young tableaux.
- “Intertwining operators” $\tau_i$ move between alcoves; dotted lines correspond to $\tau_i = 0$.

$J = \emptyset$

$J = \{\varepsilon_2 - \varepsilon_1\}$

$J = \{\varepsilon_2\}$
A little more detail

• $J$ is determined by a set of positive roots (corresp. to hyperplanes).
• For “nice” characters, there is a bijection between alcoves and marked type-C generalized Young tableaux.
• “Intertwining operators” $\tau_i$ move between alcoves; dotted lines correspond to $\tau_i = 0$.

\[ J = \emptyset \quad -1 \ 2 \ 1 \ 2 \]

\[ J = \{ \varepsilon_2 - \varepsilon_1 \} \]

\[ J = \{ \varepsilon_2 \} \quad -2 -1 \ 1 \ 2 \]

\[ J = \{ \varepsilon_2, \varepsilon_2 - \varepsilon_1 \} \]
Thm. (D.-Ram)

(1) Representations of $\mathcal{H}_k$ are indexed by pairs $(c, J)$. 
Thm. (D.-Ram)
(1) Representations of $\mathcal{H}_k$ are indexed by pairs $(c, J)$.
(2) The representations of $\mathcal{H}_k$ that factor through the Temperley-Lieb quotient are as above.
Thm. (D.-Ram)
(1) Representations of $\mathcal{H}_k$ are indexed by pairs $(c, J)$.
(2) The representations of $\mathcal{H}_k$ that factor through the Temperley-Lieb quotient are as above.
Diagrams:

Aff. type

$C'$ Hecke: