Quasisymmetric power sums

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Joint work with
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Some combinatorics

Partitions:

\[ = (5, 4, 4, 2) = \lambda \]

Compositions:

\[ = (4, 2, 5, 4) = \alpha \]
Some combinatorics

Partitions:

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\begin{array}{c}
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\text{Diagram}
\end{array}
\end{array}
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Compositions:

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\] = (4, 2, 5, 4) = \alpha

For a composition \(\alpha\),

- let \(|\alpha|\) be the size (\# boxes) of \(\alpha\);
- let \(\ell(\alpha)\) be the length (\# parts) of \(\alpha\); and
- let \(\tilde{\alpha}\) be the rearrangement of the parts of \(\alpha\) into decreasing order.

For example, \(|\alpha| = 15\), \(\ell(\alpha) = 4\), and \(\tilde{\alpha} = \lambda\).
Some combinatorics

Partitions: $(5, 4, 4, 2) = \lambda$

Compositions: $(4, 2, 5, 4) = \alpha$

For a composition $\alpha$,
- let $|\alpha|$ be the size (\# boxes) of $\alpha$;
- let $\ell(\alpha)$ be the length (\# parts) of $\alpha$; and
- let $\tilde{\alpha}$ be the rearrangement of the parts of $\alpha$ into decreasing order.

For example, $|\alpha| = 15$, $\ell(\alpha) = 4$, and $\tilde{\alpha} = \lambda$.

For compositions $\alpha$ and $\beta$, we say $\alpha$ refines $\beta$, written $\alpha \trianglelefteq \beta$, if $\beta$ can be built by combining adjacent parts of $\alpha$. For example,
Consider the complex polynomial ring in variables $x_1, x_2, \ldots, x_n$, and let $S_n$ act by permutation of the variables. Then define

$$\text{Sym}_n = \mathbb{C}[x_1, \ldots, x_n]^{S_n}.$$
Symmetric functions

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This is a graded ring, with homogeneous components

$$\text{Sym}_n^k = \{\text{homogeneous } p \in \text{Sym}_n \text{ of deg. } k\}.$$
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Think: symmetric functions in $\mathbb{C}[[x_1, x_2, \ldots]]$. 
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Think: symmetric functions in $\mathbb{C}[x_1, x_2, \ldots]$. 

Lots of favorite bases: Any basis of $\text{Sym}$ can be indexed by integer partitions $\lambda \vdash n$. 
Favorite bases of $\text{Sym}$

Monomial symmetric functions:

$$m_\lambda = \sum_{\substack{\bar{\alpha} = \lambda \\ i_1 < i_2 < \cdots < i_\ell}} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}$$
Favorite bases of Sym

Monomial symmetric functions:

\[ m_\lambda = \sum_{\bar{\alpha} = \lambda} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}, \]

Ex: \[ \square = x_1^2 x_2 + x_2^2 x_1 + x_1^2 x_3 + \cdots \]
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Monomial symmetric functions:

\[ m_\lambda = \sum_{\widetilde{\alpha} = \lambda, \ i_1 < i_2 < \cdots < i_\ell} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}, \]

Ex: \[ m_\square = x_1^2 x_2 + x_2^2 x_1 + x_1^2 x_3 + \cdots \]

Homogeneous symmetric functions:

\[ h_r = \sum_{|\alpha| = r, \ i_1 < i_2 < \cdots < i_\ell} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}, \]

\[ h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots. \]
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Monomial symmetric functions:

\[ m_\lambda = \sum_{\substack{\alpha \vdash \lambda \\text{s.t.} \\alpha_1 < \alpha_2 < \cdots < \alpha_\ell \\text{and} \\alpha_1 > 0}} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}, \]

Ex: \( m_{\square} = x_1^2 x_2 + x_2^2 x_1 + x_1^2 x_3 + \cdots \)

Homogeneous symmetric functions:

\[ h_r = \sum_{\substack{\alpha \vdash r \\text{s.t.} \\alpha_1 + \alpha_2 + \cdots + \alpha_\ell = r \\text{and} \\alpha_1 > 0}} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell} \]

\[ h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots \]

Example:

\[ h_2 = x_1^2 + x_2^2 + \cdots + x_1 x_2 + x_1 x_3 + \cdots \]
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Monomial symmetric functions:

\[ m_\lambda = \sum_{\alpha = \lambda}^{\alpha_1 < \alpha_2 < \cdots < \alpha_\ell} x^{\alpha_1}_{i_1} x^{\alpha_2}_{i_2} \cdots x^{\alpha_\ell}_{i_\ell}, \]

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Homogeneous symmetric functions:

\[ h_r = \sum_{\substack{\alpha \atop |\alpha| = r \atop i_1 < i_2 < \cdots < i_\ell}} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell} = \sum_{|\lambda| = r} m_\lambda, \]

\[ h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots. \]

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Example:

$$h_2 = x_1^2 + x_2^2 + \cdots + x_1 x_2 + x_1 x_3 + \cdots = m_\square + m_\square, \quad h_\square = h_2 h_1$$
Favorite bases of $\text{Sym}$

Monomial symmetric functions:

\[ m_\lambda = \sum_{\alpha=\lambda, \ i_1<i_2<\cdots<i_\ell} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}, \]

Ex: \( m_{\begin{array}{c} 1 \\ 2 \\ 3 \end{array}} = x_1^2 x_2 + x_2^2 x_1 + x_3^2 x_1 + \cdots \)

Homogeneous symmetric functions:

\[ h_r = \sum_{|\alpha|=r} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell} = \sum_{|\lambda|=r} m_\lambda, \quad h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots. \]

Example:

\[ h_2 = x_1^2 + x_2^2 + \cdots + x_1 x_2 + x_1 x_3 + \cdots = m_{\begin{array}{c} 1 \\ 2 \end{array}} + m_{\begin{array}{c} 1 \\ 3 \end{array}}, \]

\[ h_{\begin{array}{c} 1 \\ 2 \end{array}} = h_2 h_1 = (m_{\begin{array}{c} 1 \\ 2 \end{array}} + m_{\begin{array}{c} 1 \\ 3 \end{array}}) m_{\begin{array}{c} 1 \end{array}} = 2m_{\begin{array}{c} 1 \\ 2 \\ 3 \end{array}} + m_{\begin{array}{c} 1 \\ 3 \\ 4 \end{array}} + m_{\begin{array}{c} 2 \\ 3 \\ 4 \end{array}}. \]
Favorite bases of Sym

Monomial symmetric functions:

\[ m_\lambda = \sum_{\substack{\tilde{\alpha} = \lambda \\ i_1 < i_2 < \cdots < i_\ell}} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}, \quad \text{Ex: } m_{\square} = x_1^2 x_2 + x_2^2 x_1 + x_1^2 x_3 + \cdots \]

Homogeneous symmetric functions:

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Example:

\[ h_2 = x_1^2 + x_2^2 + \cdots + x_1 x_2 + x_1 x_3 + \cdots = m_{\square} + m_{\square}, \]

\[ h_{\square} = h_2 h_1 = (m_{\square} + m_{\square}) m_{\square} = 2m_{\square} + m_{\square \square} + m_{\square \square \square}. \]

Scalar product: \( \langle , \rangle : \text{Sym} \otimes \text{Sym} \rightarrow \mathbb{C} \) defined by

\[ \langle h_\lambda, m_\mu \rangle = \delta_{\lambda,\mu}, \]

so that the homogeneous and monomial functions are dual.
Favorite bases of Sym

Elementary symmetric functions:

\[ e_r = \sum_{1 \leq i_1 < i_2 < \cdots < i_r} x_{i_1} \cdots x_{i_r} = m_{(1,1,\ldots,1)} \]

\[ e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots \]
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For example,

$$e_2 = x_1 x_2 + x_1 x_3 + \cdots = m_{\square},$$

$$e_{\square} = e_2 e_1$$
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For example,

\[ e_2 = x_1x_2 + x_1x_3 + \cdots = m_3, \]

\[ e_{\boxed{2}} = e_2 e_1 = m_{\boxed{2}} m_{\boxed{1}} = m_{\boxed{3}} + m_{\boxed{1}}. \]
Favorite bases of \( \text{Sym} \)

Elementary symmetric functions:

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e_r = \sum_{1 \leq i_1 < i_2 < \cdots < i_r} x_{i_1} \cdots x_{i_r} = m_{(1,1,\ldots,1)} \quad e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots.
\]

For example,

\[
e_2 = x_1 x_2 + x_1 x_3 + \cdots = m_{-},
\]

\[
e_{-} = e_2 e_1 = m_{-} m_{-} = m_{-} + m_{-}.
\]

Schur functions:

\[
s_\lambda \sum_{\text{ss tabl. } T \text{ of shape } \lambda} x^\text{wt}(T) = \sum_{\mu} K_{\lambda\mu} m_{\mu},
\]

where the coefficients \( K_{\lambda\mu} \) are the Kostka numbers.
**Favorite bases of Sym**

**Elementary symmetric functions:**

\[ e_r = \sum_{1 \leq i_1 < i_2 < \ldots < i_r} x_{i_1} \cdots x_{i_r} = m_{(1,1,\ldots,1)} \quad e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots . \]

For example,

\[ e_2 = x_1 x_2 + x_1 x_3 + \cdots = m_{\square}, \]

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Note

\[ \langle s_\lambda, s_\mu \rangle = \delta_{\lambda,\mu} \quad \text{and} \quad \langle e_\lambda, \omega(m_\mu) \rangle = \delta_{\lambda,\mu} \]

where \( \omega \) is the involution on Sym sending \( e_\lambda \to h_\lambda \).
Favorite bases of Sym

Power sum symmetric functions:

\[ p_r = x_1^r + x_2^r + \cdots \quad \text{and} \quad p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots . \]
Favorite bases of Sym

Power sum symmetric functions:

\[ p_r = x_1^r + x_2^r + \cdots \quad \quad \quad p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots . \]

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\[ p_2 = x_1^2 + x_2^2 + \cdots \]

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Favorite bases of Sym

Power sum symmetric functions:

\[ p_r = x_1^r + x_2^r + \cdots = m(r), \quad p_{\lambda} = p_{\lambda_1} p_{\lambda_2} \cdots. \]

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For example,

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$$p_{\square} = p_2p_1$$
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For example,

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For example,

\[ p_2 = x_1^2 + x_2^2 + \cdots = m\square, \]

\[ p_{\square\square} = p_2 p_1 = m\square m\square = m\boxed{\square} + m\boxed{\square\square}. \]
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For example,

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\[ p_{\square} = p_2 p_1 = m_{\square} m_{\square} = m_{\square} + m_{\square \square}. \]

We have

\[ \langle p_\lambda, p_\mu \rangle = z_\lambda \delta_{\lambda \mu} \]

where \( z_\lambda \) is the size of the stabilizer of a permutation of cycle type \( \lambda \):

\[ z_\lambda = \prod_k a_k! k^{a_k}, \quad a_k = \# \{ \text{pts of length } k \} \]
Favorite bases of $\text{Sym}$

Power sum symmetric functions:

$$p_r = x_1^r + x_2^r + \cdots = m(r), \quad p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots .$$

For example,

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where $z_\lambda$ is the size of the stabilizer of a permutation of cycle type $\lambda$:

$$z_\lambda = \prod_k a_k! k^{a_k}, \quad a_k = \# \{ \text{pts of length } k \} \quad \text{Ex: } z_{\square \square \square} = 2! \ 3^2.$$
Generating functions

\[ H(t) = \sum_{k \geq 0} h_k t^k = \prod_{i \geq 1} (1 - x_i t)^{-1} \]

\[ E(t) = \sum_{k \geq 0} e_k t^k = \prod_{i \geq 1} (1 + x_i t) \]
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Note \( H(t) = 1/E(-t) \).
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Note \( H(t) = 1/E(-t). \)

\[ P(t) = \sum_{k \geq 0} p_k t^k = \frac{d}{dt} \ln(H(t)) = \frac{d}{dt} \ln(1/E(-t)) \]
Variations on Sym

The ring of noncommutative symmetric functions $\text{NSym}$ is the $\mathbb{C}$-algebra generated by the free group on $e_1, e_2, \ldots$. 
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The ring of noncommutative symmetric functions $\text{NSym}$ is the $\mathbb{C}$-algebra generated by the free group on $e_1, e_2, \ldots$.

Think: The elementary symmetric functions $e_1, e_2, \ldots$ generate $\text{Sym}$, and, aside from commuting, are algebraically independent. Now, we’re lifting to an algebra where the elementary functions no longer commute. So the abelianization

$$Ab : \text{NSym} \to \text{Sym}$$

is surjective (with kernel generated by commutators).
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Analogous bases indexed by compositions $\alpha$. 

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\[ Ab(e_\alpha) = e_{\tilde{\alpha}} \]
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- Noncom. homog.: $h_\alpha = h_{\alpha_1} \cdots h_{\alpha_\ell}$, where $h_i$ is defined by . . .

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\text{if} \quad E(t) = \sum_{k \geq 0} e_k t^k \quad \text{and} \quad H(t) = \sum_{k \geq 0} h_k t^k,
\]

\[
\text{then} \quad H(t) = 1/E(-t). \quad \text{(Recall: } H(t) = 1/E(-t) \text{ in } \text{Sym}).
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  \text{then } H(t) = 1/E(-t). \quad \text{(Recall: } H(t) = 1/E(-t) \text{ in Sym).}
  \]

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\quad Ab(h_\alpha) = h_{\tilde{\alpha}}
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- Noncom. power sums: two choices, $\psi$ and $\phi$!
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Analogous bases indexed by compositions $\alpha$.

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then $H(t) = 1/E(-t)$. (Recall: $H(t) = 1/E(-t)$ in $\text{Sym}$).

\[ \text{Ab}(h_\alpha) = h_{\tilde{\alpha}} \]

- **Noncom. power sums:** two choices, $\psi$ and $\phi$!

\[
\begin{align*}
\text{In Sym:} & \quad P(t) = \frac{d}{dt} \ln(H(t)) \quad \frac{d}{dt} H(t) = H(t) \Psi(t) \\
\text{In NSym:} & \quad \text{Type 1:} \quad P(t) = \frac{d}{dt} \ln(H(t)) \quad \frac{d}{dt} H(t) = H(t) \Psi(t)
\end{align*}
\]
The ring of noncommutative symmetric functions $\text{NSym}$ is the $\mathbb{C}$-algebra generated by the free group on $e_1, e_2, \ldots$.

Analogous bases indexed by compositions $\alpha$.

- Noncom. elementary: $e_\alpha = e_{\alpha_1} \cdots e_{\alpha_\ell}$. 
  \[ \mathcal{A}b(e_\alpha) = e_{\tilde{\alpha}} \]

- Noncom. homog.: $h_\alpha = h_{\alpha_1} \cdots h_{\alpha_\ell}$, where $h_i$ is defined by...

  \[
  \text{if } E(t) = \sum_{k \geq 0} e_k t^k \quad \text{and} \quad H(t) = \sum_{k \geq 0} h_k t^k, \\
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  \text{In } \text{Sym}: \\
  \text{Type 1: } P(t) = \frac{d}{dt} \ln(H(t)) \quad \frac{d}{dt}H(t) = H(t) \Psi(t) \\
  \text{Type 2: } H(t) = \exp \left( \int P(t) dt \right) \quad H(t) = \exp \left( \int \Phi(t) dt \right)
  \]
The ring of noncommutative symmetric functions $\text{NSym}$ is the $\mathbb{C}$-algebra generated by the free group on $e_1, e_2, \ldots$.

Analogous bases indexed by compositions $\alpha$.

- Noncom. elementary: $e_\alpha = e_{\alpha_1} \cdots e_{\alpha_\ell}$. $Ab(e_\alpha) = e_{\tilde{\alpha}}$

- Noncom. homog.: $h_\alpha = h_{\alpha_1} \cdots h_{\alpha_\ell}$, where $h_i$ is defined by...

\[
\text{if } E(t) = \sum_{k \geq 0} e_k t^k \quad \text{and} \quad H(t) = \sum_{k \geq 0} h_k t^k,
\]

then $H(t) = 1/E(-t)$. (Recall: $H(t) = 1/E(-t)$ in $\text{Sym}$).

$Ab(h_\alpha) = h_{\tilde{\alpha}}$

- Noncom. power sums: two choices, $\psi$ and $\phi$!

In $\text{Sym}$:

Type 1: $P(t) = \frac{d}{dt} \ln(H(t))$  \quad $\frac{d}{dt}H(t) = H(t)\Psi(t)$

Type 2: $H(t) = \exp \left( \int P(t) dt \right)$  \quad $H(t) = \exp \left( \int \Phi(t) dt \right)$

Not the same! (No unique notion of log derivative for power series with noncommutative coefficients.) But

$Ab(\psi_\alpha) = p_{\tilde{\alpha}} = Ab(\phi_\alpha)$
Variations on Sym

The ring of quasisymmetric functions $\mathbb{QSym}$ is a subring of $\mathbb{C}[x_1, x_2, \ldots]$ consisting of series where the coefficients on the monomials

$$x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_\ell^{\alpha_\ell} \quad \text{and} \quad x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}$$

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For example,

$$\sum_{i<j} x_ix_j^2 = x_1x_2^2 + x_1x_3^2 + x_2x_3^2 + \cdots$$

is quasisymmetric but not symmetric (the coef. on $x_1^2x_2$ is 0).
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Bases of $\mathbb{QSym}$ are also indexed by compositions. Namely, the monomial basis has a natural analog:

$$M_\alpha = \sum_{i_1 < i_2 < \cdots < i_{\ell(\alpha)}} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_{\ell}}^{\alpha_{\ell}}, \quad \text{so that} \quad m_\lambda = \sum_{\tilde{\alpha}=\lambda} M_{\tilde{\alpha}}.$$
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Extending linearly gives a natural surjective map $\text{QSym} \to \text{Sym}$. 
Dual Hopf algebras

Both NSym and QSym have Hopf algebra structures. In particular, they are dual as Hopf algebras, meaning there is a natural pairing
\[ \langle \cdot, \cdot \rangle : \text{NSym} \otimes \text{QSym} \to \mathbb{C}. \]
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\[ \langle , \rangle : \text{NSym} \otimes \text{QSym} \to \mathbb{C}. \]

Moreover, the duality is analogous to the pairing in Sym; namely

\[ \langle h_\lambda, m_\mu \rangle = \delta_{\lambda, \mu} \quad \text{in Sym} \otimes \text{Sym} \]
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Moreover, the duality is analogous to the pairing in \( \text{Sym} \); namely

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The dual to the elementary basis in \( \text{NSym} \) is the so-called *forgotten basis* of \( \text{QSym} \). There are several notions of the analog to the Schur basis in \( \text{NSym} \) and \( \text{QSym} \), paired by duality.

Also, in \( \text{Sym} \) the power sum basis is (essentially) self-dual:

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**Question:** What is dual to \( \psi \)? to \( \phi \)?
In Sym the power sum basis is (essentially) self-dual:

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Type 1

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This is equivalent to

$$h_\alpha = \sum_{\beta \preceq \alpha} \frac{1}{\pi(\beta, \alpha)} \psi_\beta,$$

where $\pi(\beta, \alpha)$ is a combinatorial statistic on the refinement $\beta \preceq \alpha$. 
**Type 1**

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\[ \psi_\alpha^* = \sum_{\beta \succeq \alpha} \frac{1}{\pi(\alpha, \beta)} M_\beta. \]
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In \text{Sym} the power sum basis is (essentially) self-dual:

\[ \langle p_{\lambda}, p_{\mu} \rangle = z_{\lambda} \delta_{\lambda \mu} \]

In \text{NSym}, the type 1 power sum basis is defined by the generating function relation

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So, the dual in \text{QSym} will satisfy

\[
\psi_{\alpha}^* = \sum_{\beta \succeq \alpha} \frac{1}{\pi(\alpha, \beta)} M_{\beta}.
\]

Define

\[
\Psi_{\alpha} = z_{\alpha} \psi_{\alpha}^*, \quad \text{so that} \quad \langle \psi_{\alpha}, \Psi_{\beta} \rangle = z_{\alpha} \delta_{\alpha \beta}.
\]
Computing coefficients

\[ \Psi_\alpha = z\tilde{\alpha} \sum_{\beta \succeq \alpha} \frac{1}{\pi(\alpha, \beta)} M_\beta. \]
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\[ \Psi_\alpha = z^\alpha \sum_{\beta \geq \alpha} \frac{1}{\pi(\alpha, \beta)} M_\beta. \]

For example, we saw that

\begin{align*}
\pi & = \begin{bmatrix}
1 & 3 & 4
\end{bmatrix} \\
\pi_{\alpha, \beta} & = \begin{bmatrix}
1 & 3 & 4
\end{bmatrix} \\
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For example, we saw that

First, for each block, we compute the product of the partial sums:

\[
\pi \left( \begin{array}{c}
\text{refines}
\end{array} \right) = \left| \begin{array}{c}
\end{array} \right| \cdot \left| \begin{array}{c}
\end{array} \right| \cdot \left| \begin{array}{c}
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Then, for \( \alpha \) refining \( \beta \), the coefficient of \( M_\beta \) in \( \psi_\alpha^* \) is \( 1/\pi(\alpha, \beta) \), where

\[ \pi \left( \begin{array} \hline \hline \end{array} \right) = \pi \left( \begin{array} \hline \hline \end{array} \right) \pi \left( \begin{array} \hline \hline \end{array} \right) \pi \left( \begin{array} \hline \hline \end{array} \right) \pi \left( \begin{array} \hline \hline \end{array} \right) = (1 \cdot 3 \cdot 4)(2)(5)(1 \cdot 2 \cdot 4) \]
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Then, for \( \alpha \) refining \( \beta \), the coefficient of \( M_\beta \) in \( \psi_\alpha^* \) is \( \frac{1}{\pi(\alpha, \beta)} \), where

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As another example, \( z = 2 \),

\[\Psi = z \psi^* = 2 \left( \frac{1}{2} M + \frac{1}{3} M \right)\]

\[\Psi = z \psi^* = 2 \left( \frac{1}{2} M + \frac{1}{6} M \right)\]
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As another example, \( z_\square = 2 \),

\[ \Psi_\square = 2 \left( \frac{1}{2} M_\square + \frac{1}{3} M_\square \right) \]

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So

\[ \Psi_\square + \Psi_\square = M_\square + M_\square + M_\square \]
Computing coefficients

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So

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Theorem (BDHMN)

Type 1 QSym powers sum to Sym powers:

$$p_\lambda = \sum_{\tilde{\alpha} = \lambda} \Psi_\alpha.$$
Theorem: \[ p_\lambda = \sum_{\tilde{\alpha} = \lambda} \Psi_\alpha, \quad \text{where} \quad \Psi_\alpha = z\tilde{\alpha} \sum_{\alpha \preceq \beta} \frac{1}{\pi(\alpha, \beta)} M_\beta. \]

Proof outline: For compositions \( \alpha \) and \( \beta \), define \( O_{\alpha, \beta} \) be the set of ordered set partitions \( (B_1, \cdots, B_{\ell(\beta)}) \) of \( \{1, \cdots, \ell(\alpha)\} \) satisfying

\[
\beta_j = \sum_{i \in B_j} \alpha_i \quad \text{for} \quad 1 \leq j \leq \ell(\beta).
\]
Theorem: \( p_\lambda = \sum_{\tilde{\alpha} = \lambda} \Psi_\alpha, \) where \( \Psi_\alpha = z_{\tilde{\alpha}} \sum_{\alpha \preceq \beta} \frac{1}{\pi(\alpha, \beta)} M_\beta. \)

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\[
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\]

For example, if

\[
\alpha = \begin{array}{|c|c|}
\hline
0 & 1 \\
\hline
\end{array} \quad \text{and} \quad \beta = \begin{array}{|c|c|c|}
\hline
0 & 0 & 1 \\
\hline
\end{array},
\]

then \( O_{\alpha, \beta} \) contains \( (\{1, 3\}, \{2\}) \) and \( (\{2\}, \{1, 3\}) \).
Theorem: \( p_\lambda = \sum_{\tilde{\alpha} = \lambda} \Psi_\alpha \), where \( \Psi_\alpha = z\tilde{\alpha} \sum_{\alpha \leq \beta} \frac{1}{\pi(\alpha, \beta)} M_\beta \).

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For example, if

\[
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& & \\
& & \\
\end{array} \quad \text{and} \quad \beta = \begin{array}{ccc}
& & \\
& & \\
\end{array},
\]

then \( O_{\alpha, \beta} \) contains \((\{1, 3\}, \{2\})\) and \((\{2\}, \{1, 3\})\).

It has been shown that

\[
p_\lambda = \sum_{\text{part'\,n } \mu} |O_{\lambda, \mu}| m_\mu, \quad \text{so that} \quad p_\lambda = \sum_{\text{comp } \beta} |O_{\lambda, \beta}| M_\beta.
\]
Theorem: \[ p_\lambda = \sum_{\tilde{\alpha} = \lambda} \Psi_\alpha, \text{ where } \Psi_\alpha = z\tilde{\alpha} \sum_{\alpha \lessdot \beta} \frac{1}{\pi(\alpha, \beta)} M_\beta. \]

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We combinatorially prove, for a fixed partition \( \lambda \) with size \( n \), and a fixed composition \( \beta \), that

\[ |O_{\lambda\beta}| \frac{n!}{z_\lambda} = \sum_{\alpha \lessdot \beta, \tilde{\alpha} = \lambda} \frac{n!}{\pi(\alpha, \beta)}, \]
Theorem: \( p_\lambda = \sum_{\tilde{\alpha} = \lambda} \Psi_\alpha, \) where \( \Psi_\alpha = z\tilde{\alpha} \sum_{\alpha \preceq \beta} \frac{1}{\pi(\alpha, \beta)} M_\beta. \)

Proof outline: For compositions \( \alpha \) and \( \beta \), define \( O_{\alpha, \beta} \) be the set of ordered set partitions \((B_1, \cdots, B_{\ell(\beta)})\) of \( \{1, \cdots, \ell(\alpha)\} \) satisfying

\[
\beta_j = \sum_{i \in B_j} \alpha_i \text{ for } 1 \leq j \leq \ell(\beta).
\]

\( O_{\lambda, \mu} \) is the set of ordered partitions of \( \lambda \), and \( O_{\lambda, \beta} \) is the set of ordered partitions of \( \lambda \) into \( \beta \). 

It has been shown that

\[
p_\lambda = \sum_{\text{part'} \mu} |O_{\lambda, \mu}| m_\mu,
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so that \( p_\lambda = \sum_{\text{comp } \beta} |O_{\lambda, \beta}| M_\beta. \)

We combinatorially prove, for a fixed partition \( \lambda \) with size \( n \), and a fixed composition \( \beta \), that

\[
|O_{\lambda, \beta}| \cdot |S^\lambda_n| = |O_{\lambda, \beta}| \frac{n!}{z\lambda} = \sum_{\alpha \preceq \beta \tilde{\alpha} = \lambda} \frac{n!}{\pi(\alpha, \beta)},
\]

where \( S^\lambda_n = \{\sigma \in S_n \text{ of cycle type } \lambda\} \).
Two ways of thinking about permutations:

- In one-line notation:

  \[ \sigma = 571423689 \]

  is the permutation sending

  \[ 1 \mapsto 5, \ 2 \mapsto 7, \ 3 \mapsto 1, \ \text{and so on}\ldots \]
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- **In cycle notation:**
  \[ \sigma = (152763)(4)(8)(9). \]

Several equivalent ways to write in cycle notation. We say \( \sigma \) is written in **standard form** if
- largest element of each cycle is last, and
- cycles ordered increasingly according to largest element

Ex: \((4)(631527)(8)(9)\)
Two ways of thinking about permutations:

- **In one-line notation:**
  \[ \sigma = 571423689 \]
  is the permutation sending
  \[ 1 \mapsto 5, \ 2 \mapsto 7, \ 3 \mapsto 1, \ \text{and so on...} \]

- **In cycle notation:**
  \[ \sigma = (152763)(4)(8)(9). \]
  Several equivalent ways to write in cycle notation. We say \( \sigma \) is written in standard form if
  - largest element of each cycle is last, and
  - cycles ordered increasingly according to largest element
  
  **Ex:** \( (4)(631527)(8)(9) \)

- Let \( \alpha \preceq \beta \) of size \( n \), and let \( \sigma \in S_n \). We say \( \sigma \) is **consistent** with \( \alpha \preceq \beta \) if...
Two ways of thinking about permutations:

- **In one-line notation:**

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  **Ex:** let \( \alpha = (1, 1, 2, 1, 3, 1) \) and \( \beta = (2, 2, 5) \)
Two ways of thinking about permutations:

- In one-line notation:
  \( \sigma = 571423689 \)
  is the permutation sending
  \( 1 \mapsto 5, \ 2 \mapsto 7, \ 3 \mapsto 1, \) and so on...

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**Ex:** let \( \alpha = (1, 1, 2, 1, 3, 1) \) and \( \beta = (2, 2, 5) \)

Start in one-line notation: \( 571423689 \)
In cycle notation:

\[ \sigma = (152763)(4)(8)(9). \]

Several equivalent ways to write in cycle notation. We say \( \sigma \) is written in standard form if

- largest element of each cycle is last, and
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Let \( \alpha \preceq \beta \) of size \( n \), and let \( \sigma \in S_n \). We say \( \sigma \) is consistent with \( \alpha \preceq \beta \) if...

Ex: let \( \alpha = (1, 1, 2, 1, 3, 1) \) and \( \beta = (2, 2, 5) \)

Start in one-line notation:

571423689

Split according to \( \beta \):

57\|14\|23689
In cycle notation:

\[ \sigma = (152763)(4)(8)(9). \]

Several equivalent ways to write in cycle notation. We say \( \sigma \) is written in standard form if

- largest element of each cycle is last, and
- cycles ordered increasingly according to largest element

Ex: \( (4)(631527)(8)(9) \)

Let \( \alpha \preceq \beta \) of size \( n \), and let \( \sigma \in S_n \). We say \( \sigma \) is consistent with \( \alpha \preceq \beta \) if...

Ex: let \( \alpha = (1, 1, 2, 1, 3, 1) \) and \( \beta = (2, 2, 5) \)

Start in one-line notation: \( 571423689 \)

Split according to \( \beta \): \( 57|14|23689 \)

Add parentheses according to \( \alpha \): \( (5)(7)|(14)|(2)(368)(9) \)
In cycle notation:

\[ \sigma = (152763)(4)(8)(9). \]

Several equivalent ways to write in cycle notation. We say \( \sigma \) is written in standard form if

- largest element of each cycle is last, and
- cycles ordered increasingly according to largest element

Ex: \((4)(631527)(8)(9)\)

Let \( \alpha \preceq \beta \) of size \( n \), and let \( \sigma \in S_n \). We say \( \sigma \) is consistent with \( \alpha \preceq \beta \) if...

Ex: let \( \alpha = (1, 1, 2, 1, 3, 1) \) and \( \beta = (2, 2, 5) \)

Start in one-line notation: \( 571423689 \)

Split according to \( \beta \): \( 57\|14\|23689 \)

Add parentheses according to \( \alpha \): \( (5)(7)\| (14)\| (2)(368)(9) \)

If the permutations in each partition are in standard form, then \( \sigma \) is consistent.
In cycle notation:

\[ \sigma = (152763)(4)(8)(9). \]

Several equivalent ways to write in cycle notation. We say \( \sigma \) is written in standard form if

1. largest element of each cycle is last, and
2. cycles ordered increasingly according to largest element

Ex: \((4)(631527)(8)(9)\)

Let \( \alpha \preceq \beta \) of size \( n \), and let \( \sigma \in S_n \). We say \( \sigma \) is consistent with \( \alpha \preceq \beta \) if...

Ex: let \( \alpha = (1, 1, 2, 1, 3, 1) \) and \( \beta = (2, 2, 5) \)

Start in one-line notation: \( 571423689 \)

Split according to \( \beta \): \( 57\|14\|23689 \)

Add parentheses according to \( \alpha \): \( (5)(7)\| (14)\| (2)(368)(9) \)

If the permutations in each partition are in standard form, then \( \sigma \) is consistent.

Non-example: \( 571428369 \quad \rightarrow \quad (5)(7)\| (14)\| (2)(836)(9) \)
\[
\text{Cons}_{(1,2,1)} \approx_{(1,2,1)} = \{1234, 1243, 1342, 2134, 2143, 2341, 3124, \\
3142, 3241, 4123, 4132, 4231\},
\]

\[
\text{Cons}_{(1,2,1)} \approx_{(1,3)} = \{1234, 2134, 3124, 4123\},
\]

\[
\text{Cons}_{(1,2,1)} \approx_{(3,1)} = \{1234, 1243, 1342, 2134, 2143, 2341, 3142, 3241\},
\]

\[
\text{Cons}_{(1,2,1)} \approx_{(4)} = \{1234, 2134\}
\]
Cons\((1,2,1)\preceq(1,2,1)\) = \{1234, 1243, 1342, 2134, 2143, 2341, 3124, 3142, 3241, 4123, 4132, 4231\},

Cons\((1,2,1)\preceq(1,3)\) = \{1234, 2134, 3124, 4123\},

Cons\((1,2,1)\preceq(3,1)\) = \{1234, 1243, 1342, 2134, 2143, 2341, 3142, 3241\},

Cons\((1,2,1)\preceq(4)\) = \{1234, 2134\}

**Lemma**

*Fix \(\alpha \preceq \beta\) of size \(n\) Then*

\[n! = |\text{Cons}_{\alpha \preceq \beta}| \cdot \pi(\alpha, \beta).\]
Cons\((1,2,1)\not\leq (1,2,1)\) = \{1234, 1243, 1342, 2134, 2143, 2341, 3124, 3142, 3241, 4123, 4132, 4231\},

\[\pi((1, 2, 1), (1, 2, 1)) = 2\]

Cons\((1,2,1)\not\leq (1,3)\) = \{1234, 2134, 3124, 4123\},

\[\pi((1, 2, 1), (1, 3)) = 2 \cdot 3\]

Cons\((1,2,1)\not\leq (3,1)\) = \{1234, 1243, 1342, 2134, 2143, 2341, 3142, 3241\},

\[\pi((1, 2, 1), (3, 1)) = 1 \cdot 3\]

Cons\((1,2,1)\not\leq (4)\) = \{1234, 2134\}

\[\pi((1, 2, 1), (4)) = 1 \cdot 3 \cdot 4\]

**Lemma**

*Fix \(\alpha \not\leq \beta\) of size \(n\) Then*

\[n! = |Con_{\alpha \not\leq \beta}| \cdot \pi(\alpha, \beta)\]
Lemma

Fix \( \alpha \preceq \beta \) of size \( n \) Then

\[ n! = |\text{Cons}_{\alpha \preceq \beta}| \cdot \pi(\alpha, \beta). \]

Proof: Let

\[ A_{\alpha \preceq \beta} = \bigotimes_{i=1}^{\ell(\beta)} \left( \bigotimes_{j=1}^{\ell(\alpha^{(i)})} \mathbb{Z}/a_j^{(i)}\mathbb{Z} \right), \quad \text{where} \quad a_j^{(i)} = \sum_{r=1}^{j} \alpha_r^{(i)}, \]

so that \( |A_{\alpha \preceq \beta}| = \pi(\alpha, \beta) \).
Lemma

Fix $\alpha \trianglelefteq \beta$ of size $n$ Then

$$n! = |\text{Cons}_{\alpha \trianglelefteq \beta}| \cdot \pi(\alpha, \beta).$$

Proof: Let

$$A_{\alpha \trianglelefteq \beta} = \bigotimes_{i=1}^{\ell(\beta)} \left( \bigotimes_{j=1}^{\ell(\alpha^{(i)})} \mathbb{Z}/a^{(i)}_{j}\mathbb{Z} \right),$$

where $a^{(i)}_{j} = \sum_{r=1}^{j} \alpha^{(i)}_{r}$,

so that $|A_{\alpha \trianglelefteq \beta}| = \pi(\alpha, \beta)$. Then there is a bijection

$$S_n \rightarrow \text{Cons}_{\alpha \trianglelefteq \beta} \times A_{\alpha \trianglelefteq \beta} \ldots$$
Lemma

Fix $\alpha \preceq \beta$ of size $n$ Then

$$n! = |\text{Cons}_{\alpha \preceq \beta}| \cdot \pi(\alpha, \beta).$$

Proof: Let

$$A_{\alpha \preceq \beta} = \bigotimes_{i=1}^{\ell(\beta)} \left( \bigotimes_{j=1}^{\ell(\alpha^{(i)})} \mathbb{Z}/a_{j}^{(i)} \mathbb{Z} \right), \quad \text{where } a_{j}^{(i)} = \sum_{r=1}^{j} \alpha_{r}^{(i)},$$

so that $|A_{\alpha \preceq \beta}| = \pi(\alpha, \beta)$. Then there is a bijection

$$S_n \rightarrow \text{Cons}_{\alpha \preceq \beta} \times A_{\alpha \preceq \beta} \ldots$$

Example: $\alpha = (2, 3, 2, 2), \beta = (5, 4), \sigma = 739628451 \ (\in \text{Cons}_{\alpha \preceq \beta})$.

Split $\sigma$ according to $\beta$: $\underbrace{73962}_\sigma^{(1)} \| \underbrace{8451}_\sigma^{(1)}$

For each $i$, “rotate” $\sigma^{(i)}$ into consistency with to $\alpha \preceq \beta$, and record rotations...
Then there is a bijection
\[ S_n \to \text{Cons}_{\alpha \lesssim \beta} \times A_{\alpha \lesssim \beta} : \]

Example: \( \alpha = \begin{array}{c}
\hline
\hline
\hline
\end{array}, \quad \beta = \begin{array}{c}
\hline
\hline
\hline
\end{array}, \quad \sigma = 739628451 \, (\in \text{Cons}_{\alpha \lesssim \beta}) .
\]

Split \( \sigma \) according to \( \beta \): \( \underbrace{73962}_{\sigma^{(1)}} \| \underbrace{8451}_{\sigma^{(2)}} \)

For each \( i \), “rotate” \( \sigma^{(i)} \) into consistency with to \( \alpha \lesssim \beta \), and record rotations. . .

\( i = 1 : \sigma^{(1)} = 73962, \quad \beta_1 \) parts of \( \alpha \): \begin{array}{c}
\hline
\hline
\hline
\end{array}
Then there is a bijection

\[ S_n \to \text{Cons}_{\alpha \preceq \beta} \times A_{\alpha \preceq \beta} : \]

Example: \( \alpha = \begin{array}{cccc} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \end{array} \), \( \beta = \begin{array}{cccc} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \end{array} \), \( \sigma = 739628451 \ (\in \text{Cons}_{\alpha \preceq \beta}) \).

Split \( \sigma \) according to \( \beta \): \( \sigma^{(1)} \parallel \sigma^{(2)} \)

\( \sigma^{(1)} = 73962 \), \( \sigma^{(2)} = 8451 \).

For each \( i \), “rotate” \( \sigma^{(i)} \) into consistency with to \( \alpha \preceq \beta \), and record rotations...

\( i = 1 \): \( \sigma^{(1)} = 73962 \), \( \beta_1 \) parts of \( \alpha \): \begin{array}{cccc} \hline & & & \\ \hline & & & \\ \hline & & & \end{array} \)

block:
Then there is a bijection

\[ S_n \to \text{Cons}_{\alpha \preceq \beta} \times A_{\alpha \preceq \beta} : \]

**Example:** \( \alpha = \begin{array}{c}
\text{Block 1} \\
\text{Block 2} \\
\end{array} \), \( \beta = \begin{array}{c}
\text{Block 1} \\
\text{Block 2} \\
\end{array} \), \( \sigma = 739628451 \ (\in \text{Cons}_{\alpha \preceq \beta}) \).

Split \( \sigma \) according to \( \beta \):

\[ \underbrace{73962}_{\sigma^{(1)}} \| \underbrace{8451}_{\sigma^{(2)}} \]

For each \( i \), “rotate” \( \sigma^{(i)} \) into consistency with to \( \alpha \preceq \beta \), and record rotations...

\[ i = 1: \ \sigma^{(1)} = 73962, \quad \beta_1 \text{ parts of } \alpha: \begin{array}{c}
\text{Block 1} \\
\text{Block 2} \\
\end{array} \\
\text{Block: } 73962 \]
Then there is a bijection

\[ S_n \rightarrow \text{Cons}_{\alpha \preceq \beta} \times A_{\alpha \preceq \beta} : \]

Example: \( \alpha = \begin{array}{c}
\hline
\hline
\end{array} \), \( \beta = \begin{array}{c}
\hline
\hline
\end{array} \), \( \sigma = 739628451 \ (\in \text{Cons}_{\alpha \preceq \beta}) \).

Split \( \sigma \) according to \( \beta \):

\[ \begin{array}{c}
\hline
\hline
\end{array} \parallel \begin{array}{c}
\hline
\hline
\end{array} \\
(\sigma^{(1)}, \sigma^{(2)})
\]

For each \( i \), “rotate” \( \sigma^{(i)} \) into consistency with to \( \alpha \preceq \beta \), and record rotations...

\( i = 1 \):

\( \sigma^{(1)} = 73962 \), \( \beta_1 \) parts of \( \alpha \):

\[ \begin{array}{c}
\hline
\hline
\end{array} \]

block: \( 73962 \xrightarrow{\text{rotate left by 3}} 62739 \), \( s^{(1)}_2 = 3 \)
Then there is a bijection

\[ S_n \rightarrow \text{Cons}_{\alpha \lessdot \beta} \times A_{\alpha \lessdot \beta} : \]

Example: \( \alpha = \begin{array}{c}
\end{array} \), \( \beta = \begin{array}{c}
\end{array} \), \( \sigma = 739628451 \) (\( \in \text{Cons}_{\alpha \lessdot \beta} \)).

Split \( \sigma \) according to \( \beta \):

\[ \underbrace{73962}_{\sigma^{(1)}} \parallel \underbrace{8451}_{\sigma^{(2)}} \]

For each \( i \), “rotate” \( \sigma^{(i)} \) into consistency with to \( \alpha \lessdot \beta \), and record rotations...

\( i = 1 \): \( \sigma^{(1)} = 73962 \), \( \beta_1 \) parts of \( \alpha \):

\begin{array}{c}
\end{array}

block: \( 73962 \xrightarrow{\text{rotate left by 3}} 62739 \), \( s_2^{(1)} = 3 \)

\begin{array}{c}
\end{array}

block:
Then there is a bijection

$$S_n \to \text{Cons}_{\alpha \preceq \beta} \times A_{\alpha \preceq \beta} :$$

Example: \(\alpha = \begin{array}{ccc}
\text{6} & \text{3} & \text{9} \\
\text{9} & \text{6} & \text{2}
\end{array}, \quad \beta = \begin{array}{ccc}
\text{3} & \text{6} & \text{2} \\
\text{4} & \text{5} & \text{1}
\end{array}, \quad \sigma = 739628451 \ (\in \text{Cons}_{\alpha \preceq \beta}).$$

Split \(\sigma\) according to \(\beta\): \(\underbrace{73962}_{\sigma^{(1)}} \| \underbrace{8451}_{\sigma^{(2)}}\)

For each \(i\), “rotate” \(\sigma^{(i)}\) into consistency with to \(\alpha \preceq \beta\), and record rotations...

\(i = 1: \ \sigma^{(1)} = 73962, \quad \beta_1 \ \text{parts of} \ \alpha: \begin{array}{ccc}
\text{6} & \text{3} & \text{9} \\
\text{9} & \text{6} & \text{2}
\end{array}\)

block: \(73962 \xrightarrow{\text{rotate left by 3}} 62739, \quad s_{2}^{(1)} = 3\)

block: \(62|739\)
Then there is a bijection
\[ S_n \to \text{Cons}_{\alpha \preceq \beta} \times A_{\alpha \preceq \beta} : \]

Example: \( \alpha = \begin{array}{c} \hline \hline \hline \hline \end{array} \), \( \beta = \begin{array}{c|c|c|c} \hline & & & \hline & & & \hline \end{array} \), \( \sigma = 739628451 (\in \text{Cons}_{\alpha \preceq \beta}) \).

Split \( \sigma \) according to \( \beta \): \( \underbrace{73962}_{\sigma^{(1)}} \parallel \underbrace{8451}_{\sigma^{(2)}} \)

For each \( i \), “rotate” \( \sigma^{(i)} \) into consistency with to \( \alpha \preceq \beta \), and record rotations...

\( i = 1 \): \( \sigma^{(1)} = 73962 \), \( \beta_1 \) parts of \( \alpha \): \begin{array}{c|c} \hline & \hline & \hline & \hline \end{array} \)

\begin{array}{c|c|c|c} \hline & & & \hline & & & \hline \end{array} \)

\begin{array}{c|c|c|c} \hline & & & \hline & & & \hline \end{array} \)

\begin{array}{c|c|c|c|c} \hline & & & & \hline & & & & \hline \end{array} \)

\begin{array}{c|c|c|c|c} \hline & & & & \hline & & & & \hline \end{array} \)

\begin{array}{c|c|c|c|c} \hline & & & & \hline & & & & \hline \end{array} \)

\begin{array}{c|c|c|c|c} \hline & & & & \hline & & & & \hline \end{array} \)

\begin{array}{c|c|c|c|c} \hline & & & & \hline & & & & \hline \end{array} \)

\begin{array}{c|c|c|c|c} \hline & & & & \hline & & & & \hline \end{array} \)

\begin{array}{c|c|c|c|c} \hline & & & & \hline & & & & \hline \end{array} \)

\begin{array}{c|c|c|c|c} \hline & & & & \hline & & & & \hline \end{array} \)

\begin{array}{c|c|c|c|c} \hline & & & & \hline & & & & \hline \end{array} \)

\begin{array}{c|c|c|c|c} \hline & & & & \hline & & & & \hline \end{array} \)

\begin{array}{c|c|c|c|c} \hline & & & & \hline & & & & \hline \end{array} \)

\begin{array}{c|c|c|c|c} \hline & & & & \hline & & & & \hline \end{array} \)

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\begin{array}{c|c|c|c|c} \hline & & & & \hline & & & & \hline \end{array} \)

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\begin{array}{c|c|c|c|c} \hline & & & & \hline & & & & \hline \end{array} \)

\begin{array}{c|c|c|c|c} \hline & & & & \hline & & & & \hline \end{array} \)

\begin{array}{c|c|c|c|c} \hline & & & & \hline & & & & \hline \end{array} \)

\begin{array}{c|c|c|c|c} \hline & & & & \hline & & & & \hline \end{array} \)

\begin{array}{c|c|c|c|c} \hline & & & & \hline & & & & \hline \end{array} \)

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\begin{array}{c|c|c|c|c} \hline & & & & \hline & & & & \hline \end{array} \)

\begin{array}{c|c|c|c|c} \hline & & & & \hline & & & & \hline \end{array} \)

\begin{array}{c|c|c|c|c} \hline & & & & \hline & & & & \hline \end{array} \)

\begin{array}{c|c|c|c|c} \hline & & & & \hline & & & & \hline \end{array} \)

\begin{array}{c|c|c|c|c} \hline & & & & \hline & & & & \hline \end{array} \)
Then there is a bijection

$$S_n \rightarrow \text{Cons}_{\alpha \preceq \beta} \times \text{A}_{\alpha \preceq \beta} :$$

Example: $\alpha = \begin{array}{c|c|c|c|c|c}
1 & 2 & 3 & 4 & 5 & 6 \\
\hline
6 & 5 & 4 & 3 & 2 & 1
\end{array}$, $\beta = \begin{array}{c|c|c|c|c|c}
1 & 2 & 3 & 4 & 5 & 6 \\
\hline
6 & 5 & 4 & 3 & 2 & 1
\end{array}$, $\sigma = 739628451$ ($\in \text{Cons}_{\alpha \preceq \beta}$).

Split $\sigma$ according to $\beta$: $73962 \parallel 8451$

For each $i$, “rotate” $\sigma^{(i)}$ into consistency with to $\alpha \preceq \beta$, and record rotations.

$i = 1$: $\sigma^{(1)} = 73962$, $\beta_1$ parts of $\alpha$:

[[[1][1][1][1][1][1]]] block: $73962 \xrightarrow{\text{rotate left by } 3} 62739$, $s_2^{(1)} = 3$

[[[2][2][2][2]]] block: $62|739 \xrightarrow{\text{rotate left by } 2} 26|739$, $s_1^{(1)} = 2$

$i = 2$: $\sigma^{(2)} = 8451$, $\beta_2$ parts of $\alpha$: [[[[2][2][2][2]]]]
Then there is a bijection

\[ S_n \rightarrow \text{Cons}_{\alpha \preceq \beta} \times A_{\alpha \preceq \beta} : \]

Example: \( \alpha = \), \( \beta = \), \( \sigma = 739628451 (\in \text{Cons}_{\alpha \preceq \beta}) \).

Split \( \sigma \) according to \( \beta \):

\[
\sigma(1) \parallel \sigma(2)
\]

For each \( i \), “rotate” \( \sigma^{(i)} \) into consistency with to \( \alpha \preceq \beta \), and record rotations...

\( i = 1 \): \( \sigma^{(1)} = 73962 \), \( \beta_1 \) parts of \( \alpha \):

block: \( 73962 \xrightarrow{\text{rotate left by 3}} 62739 \), \( s_2^{(1)} = 3 \)

block: \( 62|739 \xrightarrow{\text{rotate left by 2}} 26|739 \), \( s_1^{(1)} = 2 \)

\( i = 2 \): \( \sigma^{(2)} = 8451 \), \( \beta_2 \) parts of \( \alpha \):

block:
Then there is a bijection
\[ S_n \rightarrow \text{Cons}_{\alpha \preceq \beta} \times A_{\alpha \preceq \beta} : \]

Example: \( \alpha = \begin{array}{c}
\text{block: 5}
\end{array} \), \( \beta = \begin{array}{c}
\text{block: 4}
\end{array} \), \( \sigma = 739628451 \ (\in \text{Cons}_{\alpha \preceq \beta}) \).

Split \( \sigma \) according to \( \beta \): \( \underbrace{73962}_{\sigma^{(1)}} \| \underbrace{8451}_{\sigma^{(2)}} \)

For each \( i \), “rotate” \( \sigma^{(i)} \) into consistency with to \( \alpha \preceq \beta \), and record rotations. . .

\( i = 1 \): \( \sigma^{(1)} = 73962 \), \( \beta_1 \) parts of \( \alpha \):

\begin{array}{c}
\text{block: 73962 \rotateleft{3} \rightarrow 62739}, \quad s_2^{(1)} = 3
\end{array}

\begin{array}{c}
\text{block: 62|739 \rotateleft{2} \rightarrow 26|739}, \quad s_1^{(1)} = 2
\end{array}

\( i = 2 \): \( \sigma^{(2)} = 8451 \), \( \beta_2 \) parts of \( \alpha \):

\begin{array}{c}
\text{block: 8451}
\end{array}
Then there is a bijection
\[ S_n \rightarrow \text{Cons}_{\alpha \preceq \beta} \times A_{\alpha \preceq \beta} : \]

Example: \( \alpha = \begin{array}{c}
\text{block:}
\end{array} \)
\[ \begin{array}{c}
\end{array} \]
\( \beta = \begin{array}{c}
\text{block:}
\end{array} \)
\[ \begin{array}{c}
\end{array} \]
\( \sigma = 739628451 \ (\in \text{Cons}_{\alpha \preceq \beta}) . \)

Split \( \sigma \) according to \( \beta \):
\[ 73962 \ parallel 8451 \]
\[ \sigma(1) \ parallel \sigma(2) \]

For each \( i \), “rotate” \( \sigma^{(i)} \) into consistency with to \( \alpha \preceq \beta \), and record rotations. . .

\( i = 1 \): \( \sigma^{(1)} = 73962 \), \( \beta_1 \) parts of \( \alpha \):
\[ \begin{array}{c}
\text{block:}
\end{array} \]
\[ 73962 \ \text{rotate left by} \ 3 \rightarrow \ 62739, \quad s_2^{(1)} = 3 \]

\[ \begin{array}{c}
\text{block:}
\end{array} \]
\[ 62 | 739 \ \text{rotate left by} \ 2 \rightarrow \ 26 | 739, \quad s_1^{(1)} = 2 \]

\( i = 2 \): \( \sigma^{(2)} = 8451 \), \( \beta_2 \) parts of \( \alpha \):
\[ \begin{array}{c}
\text{block:}
\end{array} \]
\[ 8451 \ \text{rotate left by} \ 1 \rightarrow \ 4518, \quad s_2^{(2)} = 1 \]

Invertible!
Then there is a bijection

\[ S_n \to \text{Cons}_{\alpha \succeq \beta} \times A_{\alpha \succeq \beta} : \]

Example: \( \alpha = \begin{array}{c}
\hline
1 & 2 \\
\hline
3 & 4 \\
\hline
\end{array} \), \( \beta = \begin{array}{c}
\hline
4 & 3 \\
\hline
2 & 1 \\
\hline
\end{array} \), \( \sigma = 739628451 \ (\in \text{Cons}_{\alpha \succeq \beta}) \).

Split \( \sigma \) according to \( \beta \): \( \underbrace{73962}_{\sigma^{(1)}} \| \underbrace{8451}_{\sigma^{(2)}} \)

For each \( i \), “rotate” \( \sigma^{(i)} \) into consistency with to \( \alpha \preceq \beta \), and record rotations...

\[ i = 1: \sigma^{(1)} = 73962, \quad \beta_1 \text{ parts of } \alpha: \begin{array}{c}
\hline
1 & 2 \\
\hline
3 & 4 \\
\hline
\end{array} \]

\begin{itemize}
  \item \( \box{\begin{array}{c}
\hline
1 & 2 \\
\hline
3 & 4 \\
\hline
\end{array}} \) block: \( 73962 \xrightarrow{\text{rotate left by 3}} 62739 \), \( s_2^{(1)} = 3 \)
  \item \( \box{\begin{array}{c}
\hline
1 & 2 \\
\hline
3 & 4 \\
\hline
\end{array}} \) block: \( 62|739 \xrightarrow{\text{rotate left by 2}} 26|739 \), \( s_1^{(1)} = 2 \)
\end{itemize}

\[ i = 2: \sigma^{(2)} = 8451, \quad \beta_2 \text{ parts of } \alpha: \begin{array}{c}
\hline
1 & 2 \\
\hline
3 & 4 \\
\hline
\end{array} \]

\begin{itemize}
  \item \( \box{\begin{array}{c}
\hline
1 & 2 \\
\hline
3 & 4 \\
\hline
\end{array}} \) block: \( 8451 \xrightarrow{\text{rotate left by 1}} 4518 \), \( s_2^{(2)} = 1 \)
  \item \( \box{\begin{array}{c}
\hline
1 & 2 \\
\hline
3 & 4 \\
\hline
\end{array}} \) block:
Then there is a bijection

\[ S_n \rightarrow \text{Cons}_{\alpha \preceq \beta} \times A_{\alpha \preceq \beta} : \]

Example: \( \alpha = \begin{array}{c} \ast \end{array}, \beta = \begin{array}{c} \ast \ \ast \ \ast \end{array}, \sigma = 739628451 \ (\in \text{Cons}_{\alpha \preceq \beta}). \)

Split \( \sigma \) according to \( \beta \):

\( \sigma = \begin{array}{c} 73962 \ \ | \ \ 8451 \end{array} \)

\( \sigma^{(1)} \) \( \ | \ \ \sigma^{(2)} \)

For each \( i \), “rotate” \( \sigma^{(i)} \) into consistency with to \( \alpha \preceq \beta \), and record rotations . . .

\( i = 1: \sigma^{(1)} = 73962, \ \beta_1 \) parts of \( \alpha \):

\[
\begin{array}{c}
\begin{array}{c}
\ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast \ \ast
\end{array}
\]
Then there is a bijection

\[ S_n \rightarrow \text{Cons}_{\alpha \preceq \beta} \times A_{\alpha \preceq \beta} : \]

Example: \( \alpha = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\hline \end{array} \), \( \beta = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\hline \end{array} \), \( \sigma = 739628451 \ (\in \text{Cons}_{\alpha \preceq \beta}) \).

Split \( \sigma \) according to \( \beta \):

\[ \underbrace{73962}_{\sigma^{(1)}} \bigm/ \underbrace{8451}_{\sigma^{(2)}} \]

For each \( i \), “rotate” \( \sigma^{(i)} \) into consistency with to \( \alpha \preceq \beta \), and record rotations...

\begin{align*}
\text{i} = 1: & \quad \sigma^{(1)} = 73962, \quad \beta_1 \text{ parts of } \alpha: \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\hline \end{array} \\
\text{block: } & \quad 73962 \quad \text{rotate left by 3} \quad \rightarrow \quad 62739, \quad s^{(1)}_2 = 3 \\
\text{block: } & \quad 62|739 \quad \text{rotate left by 2} \quad \rightarrow \quad 26|739, \quad s^{(1)}_1 = 2 \\
\text{i} = 2: & \quad \sigma^{(2)} = 8451, \quad \beta_2 \text{ parts of } \alpha: \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\hline \end{array} \\
\text{block: } & \quad 8451 \quad \text{rotate left by 1} \quad \rightarrow \quad 4518, \quad s^{(2)}_2 = 1 \\
\text{block: } & \quad 45|18 \quad \text{rotate left by 0} \quad \rightarrow \quad 45|18 
\end{align*}
Then there is a bijection

\[ S_n \rightarrow \text{Cons}_{\alpha \preceq \beta} \times A_{\alpha \preceq \beta} : \]

**Example:** \( \alpha = \begin{array}{c}
\hline
\hline
\hline
\hline
\end{array} \), \( \beta = \begin{array}{c}
\hline
\hline
\hline
\hline
\end{array} \), \( \sigma = 739628451 \ (\in \text{Cons}_{\alpha \preceq \beta}) \).

Split \( \sigma \) according to \( \beta \):

\[ 73962 \parallel 8451 \]

\[ \sigma^{(1)} \parallel \sigma^{(2)} \]

For each \( i \), “rotate” \( \sigma^{(i)} \) into consistency with to \( \alpha \preceq \beta \), and record rotations.

\[ i = 1: \ \sigma^{(1)} = 73962, \ \beta_1 \text{ parts of } \alpha: \begin{array}{c}
\hline
\hline
\hline
\hline
\end{array} \]

\[ \begin{array}{c}
\hline
\hline
\hline
\hline
\end{array} \text{ block: } \begin{array}{c}
73962 \text{ rotate left by } 3
\end{array} \rightarrow \begin{array}{c}
62739
\end{array}, \ s_2^{(1)} = 3 \]

\[ \begin{array}{c}
\hline
\hline
\hline
\hline
\end{array} \text{ block: } \begin{array}{c}
62|739 \text{ rotate left by } 2
\end{array} \rightarrow \begin{array}{c}
26|739
\end{array}, \ s_1^{(1)} = 2 \]

\[ i = 2: \ \sigma^{(2)} = 8451, \ \beta_2 \text{ parts of } \alpha: \begin{array}{c}
\hline
\hline
\hline
\hline
\end{array} \]

\[ \begin{array}{c}
\hline
\hline
\hline
\hline
\end{array} \text{ block: } \begin{array}{c}
8451 \text{ rotate left by } 1
\end{array} \rightarrow \begin{array}{c}
4518
\end{array}, \ s_2^{(2)} = 1 \]

\[ \begin{array}{c}
\hline
\hline
\hline
\hline
\end{array} \text{ block: } \begin{array}{c}
45|18 \text{ rotate left by } 0
\end{array} \rightarrow \begin{array}{c}
45|18
\end{array}, \ s_1^{(2)} = 0 \]
Then there is a bijection

\[ S_n \to \text{Cons}_{\alpha \preceq \beta} \times A_{\alpha \preceq \beta} : \]

**Example:** \( \alpha = \begin{array}{c}
\hline
2 & 2 & 2 \\
\hline
\end{array} \), \( \beta = \begin{array}{c}
\hline
3 & 2 & 2 \\
\hline
\end{array} \), \( \sigma = 739628451 \ (\in \text{Cons}_{\alpha \preceq \beta}) \).

Split \( \sigma \) according to \( \beta \):

\[
\begin{array}{c}
73962 \\
\hline
\end{array} \parallel \begin{array}{c}
8451 \\
\hline
\end{array} \\
\sigma^{(1)} \parallel \sigma^{(2)}
\]

For each \( i \), “rotate” \( \sigma^{(i)} \) into consistency with to \( \alpha \preceq \beta \), and record rotations.

\[
i = 1: \sigma^{(1)} = 73962, \quad \beta_1 \text{ parts of } \alpha: \begin{array}{c}
\hline
2 & 2 & 2 \\
\hline
\end{array}
\]

\[
\begin{array}{c}
\hline
2 & 2 & 2 \\
\hline
\end{array} \text{ block: } 73962 \quad \text{rotate left by 3} \rightarrow 62739, \quad s_2^{(1)} = 3
\]

\[
\begin{array}{c}
\hline
3 & 2 & 2 \\
\hline
\end{array} \text{ block: } 62|739 \quad \text{rotate left by 2} \rightarrow 26|739, \quad s_1^{(1)} = 2
\]

\[
i = 2: \sigma^{(2)} = 8451, \quad \beta_2 \text{ parts of } \alpha: \begin{array}{c}
\hline
3 & 2 & 2 \\
\hline
\end{array}
\]

\[
\begin{array}{c}
\hline
3 & 2 & 2 \\
\hline
\end{array} \text{ block: } 8451 \quad \text{rotate left by 1} \rightarrow 4518, \quad s_2^{(2)} = 1
\]

\[
\begin{array}{c}
\hline
3 & 2 & 2 \\
\hline
\end{array} \text{ block: } 45|18 \quad \text{rotate left by 0} \rightarrow 45|18, \quad s_1^{(2)} = 0
\]

So \( 739628451 \mapsto (267394518, ((2, 3), (0, 1))) \).
Then there is a bijection

$$S_n \rightarrow \text{Cons}_{\alpha \preceq \beta} \times A_{\alpha \preceq \beta}:$$

Example: $\alpha = \begin{array}{cccc}
\text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet} \\
\text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet}
\end{array}$, $\beta = \begin{array}{cccc}
\text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet} \\
\text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet}
\end{array}$, $\sigma = 739628451 \in \text{Cons}_{\alpha \preceq \beta}$.

Split $\sigma$ according to $\beta$: $\underbrace{73962}_{\sigma^{(1)}} \| \underbrace{8451}_{\sigma^{(2)}}$

For each $i$, “rotate” $\sigma^{(i)}$ into consistency with to $\alpha \preceq \beta$, and record rotations.

$i = 1$: $\sigma^{(1)} = 73962$, $\beta_1$ parts of $\alpha$: $\begin{array}{cccc}
\text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet} \\
\text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet}
\end{array}$

\begin{array}{l}
\text{block: } 73962 \xrightarrow{\text{rotate left by 3}} 62739, \quad s_2^{(1)} = 3 \\
\text{block: } 62|739 \xrightarrow{\text{rotate left by 2}} 26|739, \quad s_1^{(1)} = 2
\end{array}

$i = 2$: $\sigma^{(2)} = 8451$, $\beta_2$ parts of $\alpha$: $\begin{array}{cccc}
\text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet} \\
\text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet}
\end{array}$

\begin{array}{l}
\text{block: } 8451 \xrightarrow{\text{rotate left by 1}} 4518, \quad s_2^{(2)} = 1 \\
\text{block: } 45|18 \xrightarrow{\text{rotate left by 0}} 45|18, \quad s_1^{(2)} = 0
\end{array}

So $739628451 \mapsto (267394518, ((2, 3), (0, 1)))$. Invertible!
Lemma

Fix $\alpha \preceq \beta$ of size $n$. Then

$$n! = \left| \text{Cons}_{\alpha \preceq \beta} \right| \cdot \pi(\alpha, \beta).$$

Proof: Let

$$A_{\alpha \preceq \beta} = \bigotimes_{i=1}^{\ell(\beta)} \left( \bigotimes_{j=1}^{\ell(\alpha^{(i)})} \mathbb{Z}/a_{j}^{(i)}\mathbb{Z} \right),$$

where $a_{j}^{(i)} = \sum_{r=1}^{j} \alpha_{r}^{(i)}$, so that $\left| A_{\alpha \preceq \beta} \right| = \pi(\alpha, \beta)$. Then there is a bijection

$$S_{n} \rightarrow \text{Cons}_{\alpha \preceq \beta} \times A_{\alpha \preceq \beta}.$$
Lemma
Fix $\alpha \preceq \beta$ of size $n$ Then
\[ n! = |\text{Cons}_{\alpha \preceq \beta}| \cdot \pi(\alpha, \beta). \]

Proof: Let
\[ A_{\alpha \preceq \beta} = \prod_{i=1}^{\ell(\beta)} \left( \prod_{j=1}^{\ell(\alpha^{(i)})} \mathbb{Z}/a_j^{(i)} \mathbb{Z} \right), \quad \text{where } a_j^{(i)} = \sum_{r=1}^{j} \alpha_r^{(i)}, \]
so that $|A_{\alpha \preceq \beta}| = \pi(\alpha, \beta)$. Then there is a bijection
\[ S_n \rightarrow \text{Cons}_{\alpha \preceq \beta} \times A_{\alpha \preceq \beta}. \]
□

Lemma
Fix $\alpha \preceq \beta$ of size $n$ Then
\[ |O_{\alpha \preceq \beta}| \cdot |S_{\lambda}^\alpha| = \sum_{\alpha \preceq \beta, \tilde{\alpha} = \lambda} |\text{Cons}_{\alpha \preceq \beta}|. \]
(Similar proof.)
Lemma
Fix $\alpha \preceq \beta$ of size $n$. Then
\[ n! = |\text{Cons}_{\alpha \preceq \beta}| \cdot \pi(\alpha, \beta). \]

Lemma
Fix $\alpha \preceq \beta$ of size $n$. Then
\[ |O_{\alpha \preceq \beta}| \cdot |S^\lambda_n| = \sum_{\substack{\alpha \preceq \beta \\ \tilde{\alpha} = \lambda}} |\text{Cons}_{\alpha \preceq \beta}|. \]

(Similar proof.)

Therefore
\[ |O_{\lambda \beta}| \cdot |S^\lambda_n| = \sum_{\substack{\alpha \preceq \beta \\ \tilde{\alpha} = \lambda}} \frac{n!}{\pi(\alpha, \beta)}, \]

so that
\[ p_{\lambda} = \sum_{\text{comp } \beta} |O_{\lambda, \beta}| M_\beta = \sum_{\tilde{\alpha} = \lambda} \Psi_\alpha, \quad \text{where} \quad \Psi_\alpha = z\tilde{\alpha} \sum_{\alpha \preceq \beta} \frac{1}{\pi(\alpha, \beta)} M_\beta, \]
as desired.
Type 2

In Sym the power sum basis is (essentially) self-dual:

$$\langle p_{\lambda}, p_{\mu} \rangle = z_{\lambda} \delta_{\lambda \mu}.$$ 

In NSym, the type 2 power sum basis is defined by the generating function relation

$$H(t) = \exp \left( \int \Phi(t) dt \right)$$
Type 2

In Sym the power sum basis is (essentially) self-dual:

\[ \langle p_\lambda, p_\mu \rangle = z_\lambda \delta_{\lambda\mu}. \]

In NSym, the type 2 power sum basis is defined by the generating function relation

\[ H(t) = \exp \left( \int \Phi(t) dt \right) \]

This is equivalent to

\[ h_\alpha = \sum_{\beta \preceq \alpha} \frac{1}{\text{sp}(\beta, \alpha)} \phi_\beta, \]

where \( \text{sp}(\beta, \alpha) \) is a combinatorial statistic on the refinement \( \beta \preceq \alpha \).
Type 2

In Sym the power sum basis is (essentially) self-dual:

\[ \langle p_\lambda, p_\mu \rangle = z_\lambda \delta_{\lambda\mu}. \]

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This is equivalent to

\[ h_\alpha = \sum_{\beta \preceq \alpha} \frac{1}{\text{sp}(\beta, \alpha)} \phi_\beta, \]

where \( \text{sp}(\beta, \alpha) \) is a combinatorial statistic on the refinement \( \beta \preceq \alpha \). So, the dual in QSym will satisfy

\[ \phi^*_\alpha = \sum_{\beta \succeq \alpha} \frac{1}{\text{sp}(\alpha, \beta)} M_\beta. \]
Type 2

In Sym the power sum basis is (essentially) self-dual:

$$\langle p_\lambda, p_\mu \rangle = z_\lambda \delta_{\lambda \mu}.$$  

In NSym, the type 2 power sum basis is defined by the generating function relation

$$H(t) = \exp \left( \int \Phi(t) dt \right)$$

This is equivalent to

$$h_\alpha = \sum_{\beta \preceq \alpha} \frac{1}{\text{sp}(\beta, \alpha)} \phi_\beta,$$

where $\text{sp}(\beta, \alpha)$ is a combinatorial statistic on the refinement $\beta \preceq \alpha$. So, the dual in QSym will satisfy

$$\phi_\alpha^* = \sum_{\beta \succeq \alpha} \frac{1}{\text{sp}(\alpha, \beta)} M_\beta.$$  

Define

$$\Phi_\alpha = z_\alpha \phi_\alpha^*,$$

so that

$$\langle \phi_\alpha, \Phi_\beta \rangle = z_\alpha \delta_{\alpha \beta}.$$
Computing coefficients

\[ \Phi_\alpha = z\tilde{\alpha} \sum_{\beta \trianglerighteq \alpha} \frac{1}{\text{sp}(\alpha, \beta)} M_\beta. \]

For example, we saw that

\[ \text{sp}(\gamma) = \ell(\gamma)! \prod_{k} \gamma_j! \]

Then, for \( \alpha \) refining \( \beta \), the coefficient of \( M_\beta \) in \( \psi^* \alpha \) is \( \frac{1}{\text{sp}(\alpha, \beta)} \), where

\[ \text{sp}(\alpha, \beta) = \text{sp}(\alpha) \cdot \text{sp}(\beta) \cdot \text{sp}(\gamma) \cdot \text{sp}(\delta) \]
Computing coefficients

\[ \Phi_\alpha = z^\alpha \sum_{\beta \geq \alpha} \frac{1}{\text{sp}(\alpha, \beta)} M_\beta. \]

For example, we saw that

First, for each block, we compute \( \text{sp}(\gamma) = \ell(\gamma)! \prod_k \gamma_j : \)

\[ \text{sp} \left( \begin{array}{c}
\end{array} \right) = 3!(1 \cdot 2 \cdot 1) \]

refines
Computing coefficients

\[ \Phi_\alpha = z_\alpha \sum_{\beta \succeq \alpha} \frac{1}{\text{sp}(\alpha, \beta)} M_\beta. \]

For example, we saw that

\[ \text{sp}(\gamma) = \ell(\gamma)! \prod_k \gamma_j: \]

\[ \text{sp} \left( \begin{array}{c} \text{red} \\ \text{green} \\ \text{blue} \end{array} \right) = 3!(1 \cdot 2 \cdot 1) \]

Then, for \( \alpha \) refining \( \beta \), the coefficient of \( M_\beta \) in \( \psi_\alpha^* \) is \( 1/\text{sp}(\alpha, \beta) \), where

\[ \text{sp} \left( \begin{array}{c} \text{red} \\ \text{green} \\ \text{blue} \end{array} \right) = \text{sp} \left( \begin{array}{c} \text{red} \end{array} \right) \text{sp} \left( \begin{array}{c} \text{green} \end{array} \right) \text{sp} \left( \begin{array}{c} \text{blue} \end{array} \right) \text{sp} \left( \begin{array}{c} \text{blue} \end{array} \right) \]

\[ = 3!(1 \cdot 2 \cdot 1) \cdot 1!(2) \cdot 1!(5) \cdot 3!(1 \cdot 1 \cdot 2) \]
Computing coefficients

\[ \text{sp } \left( \begin{array}{c} \bullet \end{array} \right) = \ell(\gamma)! \prod_k \gamma_j = 3!(1 \cdot 2 \cdot 1) \]

\[ \text{sp } \left( \begin{array}{c} \bullet \end{array}, \begin{array}{c} \bullet \end{array} \right) = \text{sp } \left( \begin{array}{c} \bullet \end{array} \right) \text{sp } \left( \begin{array}{c} \bullet \end{array} \right) \text{sp } \left( \begin{array}{c} \bullet \end{array} \right) \text{sp } \left( \begin{array}{c} \bullet \end{array} \right) \]

As another example, \( z \left( \begin{array}{c} \bullet \end{array} \right) = 2, \)

\[ \Phi \left( \begin{array}{c} \bullet \end{array} \right) = z \left( \begin{array}{c} \bullet \end{array} \bullet \right) \phi^* = 2 \left( \frac{1}{2} M + \frac{1}{4} M \right) \]

\[ \Phi \left( \begin{array}{c} \bullet \end{array} \right) = z \left( \begin{array}{c} \bullet \end{array} \bullet \right) \phi^* = 2 \left( \frac{1}{2} M + \frac{1}{4} M \right) \]
Computing coefficients

\[ \text{sp} \left( \begin{array}{c}
\text{cell1} \\
\text{cell2}
\end{array} \right) = \ell(\gamma)! \prod_k \gamma_j = 3!(1 \cdot 2 \cdot 1) \]

\[ \text{sp} \left( \begin{array}{c}
\text{cell1, cell2} \\
\text{cell3, cell4}
\end{array} \right) = \text{sp} \left( \begin{array}{c}
\text{cell1}
\end{array} \right) \text{sp} \left( \begin{array}{c}
\text{cell2}
\end{array} \right) \text{sp} \left( \begin{array}{c}
\text{cell3}
\end{array} \right) \text{sp} \left( \begin{array}{c}
\text{cell4}
\end{array} \right) \]

As another example, \( z_{\text{cell1}} = 2 \),

\[ \Phi_{\text{cell1}} = z_{\text{cell1}} \phi^*_{\text{cell1}} = 2 \left( \frac{1}{2} M_{\text{cell1}} + \frac{1}{4} M_{\text{cell2}} \right) \]

\[ \Phi_{\text{cell2}} = z_{\text{cell2}} \phi^*_{\text{cell2}} = 2 \left( \frac{1}{2} M_{\text{cell1}} + \frac{1}{4} M_{\text{cell2}} \right) \]

So

\[ \Phi_{\text{cell1}} + \Phi_{\text{cell2}} = M_{\text{cell1}} + M_{\text{cell2}} + M_{\text{cell3}} \]
Computing coefficients

$$\text{sp} \left( \begin{array}{c} \vdots \\ \vdots \end{array} \right) = \ell(\gamma)! \prod_k \gamma_j = 3!(1 \cdot 2 \cdot 1)$$

$$\text{sp} \left( \begin{array}{cc} \\ & \\ \end{array} \right) = \text{sp} \left( \begin{array}{c} \vdots \\ \vdots \end{array} \right) \text{sp} \left( \begin{array}{c} \vdots \\ \vdots \end{array} \right) \text{sp} \left( \begin{array}{c} \vdots \\ \vdots \end{array} \right) \text{sp} \left( \begin{array}{c} \vdots \\ \vdots \end{array} \right)$$

As another example, $z = 2$,

$$\Phi = z \phi^* = 2 \left( \frac{1}{2} M + \frac{1}{4} M \right)$$

$$\Phi = z \phi^* = 2 \left( \frac{1}{2} M + \frac{1}{4} M \right)$$

So

$$\Phi + \Phi = M + M + M = m + m$$
Computing coefficients

\[ \text{sp} \left( \begin{array}{c} \square \\ \square \end{array} \right) = \ell(\gamma)! \prod_k \gamma_j = 3!(1 \cdot 2 \cdot 1) \]

\[ \text{sp} \left( \begin{array}{c} \square \\ \square, \square \end{array} \right) = \text{sp} \left( \begin{array}{c} \square \\ \square \end{array} \right) \text{sp} \left( \begin{array}{c} \square \\ \square \end{array} \right) \text{sp} \left( \begin{array}{c} \square \\ \square \end{array} \right) \text{sp} \left( \begin{array}{c} \square \\ \square \end{array} \right) \]

As another example, \( z \square = 2, \)

\[ \Phi \square = z \square \phi^* \square = 2 \left( \frac{1}{2} M \square + \frac{1}{4} M \square \right) \]

\[ \Phi \square = z \square \phi^* \square = 2 \left( \frac{1}{2} M \square + \frac{1}{4} M \square \right) \]

So

\[ \Phi \square + \Phi \square = M \square + M \square + M \square \]

\[ = m \square + m \square = m \square m \square = p \square p \square = p \square \square. \]
Computing coefficients

As another example, \( z = 2 \),

\[
\Phi = z \phi^* = 2 \left( \frac{1}{2} M + \frac{1}{4} M \right)
\]

So

\[
\Phi + \Phi = M + M + M
= m + m = m m = p p = p.
\]

Theorem (BDHNM)

Type 2 QSym powers sum to Sym powers:

\[
p_{\lambda} = \sum_{\tilde{\alpha}=\lambda} \Phi_{\alpha}.
\]