Centralizers of the Lie superalgebra $\mathfrak{p}(n)$, where loops go to die

Zajj Daugherty
The City College of New York
Joint with Martina Balagovic, Maria Gorelik, Iva Halacheva, Johanna Hennig, Mee Seong Im, Gail Letzter, Emily Norton, Vera Serganova, and Catharina Stroppel

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The classical Brauer algebra

The Brauer algebra $B_k(\delta)$ is the space spanned by Brauer diagrams

\[ d = \]

perfect matchings of \(\{1, \ldots, k, 1', \ldots, k'\}\)

(equivalent under isotopy), with multiplication given by vertical concatenation, subject to the relation \(\bigcirc = \delta\).
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$$d = \begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & k \\
1' & 2' & 3' & 4' & 5' & k'
\end{array}$$

perfect matchings of \{1, \ldots, k, 1', \ldots, k'\} (equivalent under isotopy), with multiplication given by vertical concatenation, subject to the relation $\bigcirc = \delta$. For example,

$$dd' = \begin{array}{cccccc}
\bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc \\
\bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc & \bigcirc
\end{array} = \delta$$
Action on tensor space

The Brauer algebra $B_k(\delta)$ is generated by

$$s_i = \begin{array}{c}
\cdots \times \cdots \\
\hline
i & i + 1
\end{array}$$

and

$$e_i = \begin{array}{c}
\cdots \ \cdots \\
\hline
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\end{array}, \quad i = 1, \ldots, k - 1,$$

with some nice relations.
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Let $V$ be a f.d. vector space, with $\beta : V \otimes V \to \mathbb{C}$ a non-degenerate symmetric (resp. skew symmetric) bilinear form on $V$, and $\beta^*$ its dual.
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$$s_i = \begin{bmatrix} 
& \cdots & \cdot & \cdots & \cdot \\
\cdot & & \cdot & & \\
\cdot & & \cdot & & \\
& & & & \\
\cdot & & & & \\
\cdot & & & & \\
\end{bmatrix} \quad \text{and} \quad e_i = \begin{bmatrix} 
& \cdots & \cdot & \cdots & \cdot \\
\cdot & & \cdot & & \\
\cdot & & \cdot & & \\
& & & & \\
\cdot & & & & \\
\cdot & & & & \\
\end{bmatrix}, \quad i = 1, \ldots, k - 1,$$

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Let $V$ be a f.d. vector space, with $\beta : V \otimes V \to \mathbb{C}$ a non-degenerate symmetric (resp. skew symmetric) bilinear form on $V$, and $\beta^*$ its dual. Then the map $B_k(\delta) \to \text{End}(V \otimes^k)$ that sends

$$s_i \mapsto \mathbf{1} \otimes^{i-1} s \otimes \mathbf{1}^{k-i-1}, \quad e_i \mapsto \mathbf{1} \otimes^{i-1} \beta^* \beta \otimes \mathbf{1}^{k-i-1},$$

where $s(u \otimes v) = v \otimes u$, is a map

$$B_k(\delta) \to \text{End}_g(V \otimes^k)$$

when $g = \mathfrak{so}(V)$ (resp. $\mathfrak{sp}(V)$), $\delta = \dim V$ (resp. $-\dim V$).
Lie superalgebras and action on tensor space (still)

Let $V = V_0 \oplus V_1$ be a $\mathbb{Z}_2$-graded vector space. For $v \in V_i$, write $\bar{v} = i$ for its degree.
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Let \( V = V_0 \oplus V_1 \) be a \( \mathbb{Z}_2 \)-graded vector space. For \( v \in V_i \), write \( \bar{v} = i \) for its degree.

Let \( \beta : V \otimes V \to \mathbb{C} \) be a nondeg., homog., bilinear form satisfying \( \beta(u, v) = (-1)^{\bar{v}\bar{u}} \beta(v, u) \) (supersymmetric).

Then

\[
g = \{ x \in \text{End}(V) \mid \beta(xu, v) + (-1)^{\bar{x}\bar{u}} \beta(v, xu) \}
\]

is a Lie superalgebra (\( \mathbb{Z}_2 \)-graded).
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The map $B_k(\delta) \to \text{End}(V \otimes^k)$ that sends
\[
s_i \mapsto 1 \otimes^{i-1} s \otimes 1^{k-i-1}, \quad e_i \mapsto 1 \otimes^{i-1} \beta^* \beta \otimes 1^{k-i-1},
\]
where $s(u \otimes v) = (-1)^{\bar{u}\bar{v}} v \otimes u$, gives
\[
B_k(\delta) \to \text{End}_\mathfrak{g}(V \otimes^k)
\]
when $\delta = \dim V_0 - \dim V_1$. 
(Kujawa-Tharp 2014) The marked Brauer algebra $B_k(\delta, \epsilon)$, $\epsilon = \pm 1$, is the space spanned by marked Brauer diagrams

d =

caps get one $\heartsuit$ each, cups get one $\blacktriangleleft$ or $\blacktriangleright$ each, no two markings at same height.

with equivalence up to isotopy except for the local relations

\[
\begin{align*}
\blacktriangleleft & = \epsilon \quad \blacktriangle > \\
\blacktriangleright & = \epsilon \quad \blacktriangledown \\
\end{align*}
\]

and

\[
\begin{align*}
\blacktriangleleft & = \epsilon \quad \blacktriangle > \\
\blacktriangleright & = \epsilon \quad \blacktriangledown \\
\end{align*}
\]

for any adjacent markings $\blacktriangleleft x$ and $\blacktriangledown y$ (meaning no markings of height between these two).
The marked Brauer algebra $B_k(\delta, \epsilon)$, $\epsilon = \pm 1$, is the space spanned by marked Brauer diagrams with equivalence up to isotopy except for the local relations for any adjacent markings $\circ x$ and $\circ y$ (meaning no markings of height between these two). Again, multiplication is given by vertical concatenation, with relations $\bigcirc = \delta$, ...
For example,

Alternatively,
\[ \square = \epsilon \quad \square = \epsilon \]

\[ \square = \epsilon \quad \square = \delta \]

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For example,
For example,
For example,
For example,

Alternatively,
\[ \epsilon = \epsilon \]
\[ \epsilon = \epsilon \]
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\[ x = \epsilon \]
\[ y = \epsilon \]
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d = \begin{tikzpicture}[baseline=-0.65ex]
    
    
    
    \end{tikzpicture}

caps get one • each,
cups get one ▲ or ◄ each,
no two markings at same height.

with equivalence up to isotopy except for the local relations

$$\begin{array}{c}
    = \epsilon \\
    = \epsilon
\end{array} \quad \text{and} \quad \begin{array}{c}
    = \epsilon \\
    = \epsilon
\end{array}$$

for any adjacent markings ◐ $x$ and ◐ $y$ (meaning no markings of height between these two). Again, multiplication is given by vertical concatenation, with relations ◐ ◐ = $\delta$, $\begin{array}{c}
    = \\
    = \\
\end{array}$ .

Note:

(1) $B_k(\delta, 1) = B_k(\delta)$.

(2) If $\epsilon = -1$, then multiplication is well-defined exactly when $\delta = 0$. 
The marked Brauer algebra $B_k(\delta, \epsilon)$ is generated by

$$s_i = \begin{array}{c}
\cdots \\
\bigotimes
\cdots
\end{array} \quad \text{and} \quad e_i = \begin{array}{c}
\cdots \\
\bigotimes
\cdots
\end{array},$$

for $i = 1, \ldots, k - 1$, with relations exactly analogous to those for the Brauer algebra, with some $\epsilon$'s. 
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& i & +1 & \\
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Back to Lie superalgebras: $V = V_0 \oplus V_1$, let $\beta : V \otimes V \to \mathbb{C}$ is a non-degenerate, homogeneous, bilinear form on $V$, and let $\mathfrak{g}$ be the corresponding $\beta$-invariant Lie superalgebra.
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$$\beta^* : \mathbb{C} \to V \otimes V \quad \text{and} \quad s : V \otimes V \to V \otimes V \quad u \otimes v \mapsto (-1)^{\bar{u}\bar{v}} v \otimes u,$$

the map

$$e_i \mapsto 1^{\otimes i-1} \otimes \beta^* \beta \otimes 1^{k-i-1}, \quad s_i \mapsto 1^{\otimes i-1} \otimes s \otimes 1^{k-i-1},$$

for $i = 1, \ldots, k - 1$, gives

$$B_k(\delta, \epsilon) \to \text{End}_\mathfrak{g}(V \otimes^k)$$

when $\delta = \dim V_0 - \dim V_1$ and $\epsilon = (-1)^{\bar{\beta}}$ [KT14].
The peculiar Lie superalgebra $\mathfrak{p}(V)$

As we saw, when $\beta$ is even, $\mathfrak{g}$ is $\mathfrak{osp}(V)$. But what about when $\beta$ is odd?
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Let $V = V_0 \oplus V_1$, and let $\beta : V \otimes V \to \mathbb{C}$ be a non-degenerate, homogeneous, odd bilinear form on $V$, and let $\mathfrak{g}$ be the corresponding $\beta$-invariant Lie superalgebra.
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Specifically, with $n = \dim V_0 = \dim V_1$,

$$\mathfrak{p}(V) \cong \left\{ \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \in \mathfrak{gl}(n|n) \mid B = B^t, C = -C^t \right\}.$$
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The representation theory of $\mathfrak{p}(V)$ is still mysterious. In particular, $B_k(0, -1)$ was first defined by Moon in 2003 to help study $\mathfrak{p}(V)$; Kujawa and Tharp aimed to push further, getting that $V \otimes^k$ decomposes into the sum of indecomposables indexed by partitions of $k, k - 2, k - 4, \cdots > 0$. 

Moon calculated the highest weight vectors for $\mathfrak{p}(V)$ in $V \otimes V$ and $V \otimes V \otimes V$ in detail. Specifically

$$0 \rightarrow L(\beta) \rightarrow \mathfrak{p}(V) \beta \rightarrow C \rightarrow 0 \quad \quad 0 \rightarrow C \beta^* \rightarrow \wedge^2 V \rightarrow L(\beta) \rightarrow 0.$$
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Specifically

$$V \otimes V = \text{Sym}^2 V \oplus \wedge^2 V,$$

where $\text{Sym}^2 V$ and $\wedge^2 V$ are both indecomposable, but not simple:

$$0 \to L(\square) \to \text{Sym}^2 V \xrightarrow{\beta} \mathbb{C} \to 0$$

$$0 \to \mathbb{C} \xrightarrow{\beta^*} \wedge^2 V \to L(\bigotimes) \to 0.$$
Jucys-Murphy elements and the Casimir

For $i < j$, let

$$s_{i,j} = \begin{array}{c}
\vdots \\
\bullet & \cdots & \bullet \\
\end{array} \quad \text{and} \quad e_{i,j} = \begin{array}{c}
\vdots \\
\bullet & \cdots & \bullet \\
\end{array} \begin{array}{c}
\bullet \quad \downarrow \\
\end{array} \begin{array}{c}
\bullet \quad \uparrow \\
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The Brauer algebra $B_k(\delta) = B_k(\delta, 1)$ has Jucys-Murphy elements

\[ x_j = c + \sum_{i=1}^{j-1} s_{i,j} - e_{i,j}, \quad c \in \mathbb{C}, \; j = 1, \ldots, k, \]

that pairwise commute (Nazarov 1996).
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\cdots \\
\cdots \\
j
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\cdots \\
j
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Action on tensor space: Let $\gamma \in U\mathfrak{g} \otimes U\mathfrak{g}$ be the split Casimir invariant, given by

$$\gamma = \sum_{b \in \Omega} b \otimes b^*,$$

where $\Omega$ is a basis of $\mathfrak{g}$, and $\{b^* \mid b \in \Omega\}$ is the dual basis w.r.t. $\beta$. 

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\[
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\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\mbox{and} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
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\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
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where \( \Omega \) is a basis of \( \mathfrak{g} \), and \( \{ b^* \mid b \in \Omega \} \) is the dual basis w.r.t. \( \beta \). Then \( \gamma \) acts on \( V \otimes V \) as as \( s_1 - e_1 \).
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where $\Omega$ is a basis of $g$, and $\{ b^* \mid b \in \Omega \}$ is the dual basis w.r.t. $\beta$. Then $\gamma$ acts on $V \otimes V$ as as $s_1 - e_1$. So the action of $x_j$ on $V \otimes^k$ is the same as that of $\sum_{i=1}^{j-1} \gamma_{i,j}$. 
Action on $M \otimes V^\otimes k$ and cyclotomic quotients

Define the degenerate affine version $B_k(\delta)$ by

$$B_k(\delta) = \mathbb{C}[y_1, \ldots, y_k] \otimes B_k(\delta) / \langle y_i\text{-relations} \rangle,$$

where relations for the $y_i$'s are those satisfied between the $x_i$'s in $B_k(\delta)$.
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$$y_j \text{ act on } M \otimes V^\otimes k \text{ by } \sum_{i=0}^{j-1} \gamma_{i,j},$$

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$$B_k(\delta) \twoheadrightarrow \text{End}_g(M \otimes V^\otimes k).$$
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$$B_k(\delta) \twoheadrightarrow \text{End}_g(M \otimes V^\otimes k).$$

Further, let $(y_1 - a_1)(y_1 - a_2) \cdots (y_1 - a_d)$ be the minimal polynomial for the action of $y_1$ on $M \otimes V$. Then for nice $M$ and $k$,

$$B_k(\delta)/\langle(y_1 - a_1)(y_1 - a_2) \cdots (y_1 - a_d)\rangle \xrightarrow{\sim} \text{End}_g(M \otimes V^\otimes k).$$
Jucys-Murphy elements for $B_k(\delta, \epsilon)$ and the sneaky Casimir

For the marked Brauer algebra,

$$x_j = c + \sum_{i=1}^{j-1} s_{i,j} - e_{i,j}, \quad c \in \mathbb{C}, \ j = 1, \ldots, k,$$

are still the Jucys-Murphy elements. So we define the degenerate affine version similarly, with $\epsilon$’s where needed,

$$B_k(\delta, \epsilon) = \mathbb{C}[y_1, \ldots, y_k] \otimes B_k(\delta, \epsilon)/\langle y_i\text{-relations}\rangle.$$
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Questions: For $B_k(0, -1)$,

(1) what tensor space do we want analogous to $M \otimes V^\otimes k$?

(2) what’s the action of the $y_i$’s?
Jucys-Murphy elements for $B_k(\delta, \epsilon)$ and the sneaky Casimir

For the marked Brauer algebra,\[ x_j = c + \sum_{i=1}^{j-1} s_{i,j} - e_{i,j}, \quad c \in \mathbb{C}, \quad j = 1, \ldots, k, \]
are still the Jucys-Murphy elements. So we define the degenerate affine version similarly, with $\epsilon$’s where needed,
\[ B_k(\delta, \epsilon) = \mathbb{C}[y_1, \ldots, y_k] \otimes B_k(\delta, \epsilon)/\langle y_i\text{-relations}\rangle. \]

Questions: For $B_k(0, -1)$,
(1) what tensor space do we want analogous to $M \otimes V \otimes k$?
(2) what’s the action of the $y_i$’s?

Start with (2): $\mathfrak{p}(V)$ has trivial center! Namely, if $\Omega$ is a basis of $\mathfrak{p}(V)$, then $\mathfrak{p}(V)$ does not contain a dual basis with respect to $\beta$. 
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In particular, considering \( \mathfrak{p}(V) \subseteq \mathfrak{gl}(V) \), then \( \{b^* \mid b \in \Omega\} \) is a basis for \( \mathfrak{p}(V)^\perp \subseteq \mathfrak{gl}(V) \).
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\[ \gamma = \sum_{b \in \Omega} b \otimes b^* \in U\mathfrak{p}(V) \otimes U\mathfrak{p}(V) \perp. \]
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Still, we can consider its action as a element of \( U\mathfrak{gl}(V) \otimes U\mathfrak{gl}(V) \), and indeed, we get

\[ \gamma_{i,j} \text{ acts on } V^\otimes k \text{ as } s_{i,j} - e_{i,j}. \]

Good start!
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Good start! But now for (1)...
What should $M$ be in $M \otimes V^\otimes k$?

**Try 1:** For the partition $\lambda$ of size $\ell$, take the indecomposable $M(\lambda)$ indexed by $\lambda$ (the one paired with $B^\lambda$ by Moon, Kujawa-Tharp) in $V^\otimes \ell$. 

Issues:
(a) In $V \otimes V$, the minimal polynomial for $\gamma$ is $(\gamma - 1)(\gamma + 1)$. So img of $B^1(0, -1)$ in $\text{End}(V \otimes V)$ (think $M = V$, $k = 1$) is at most $B^1(0, -1) / \langle (y_1 - 1)(y_1 + 1) \rangle$ (dimension 2).

(b) Non-semisimple actions! In $V \otimes V = \text{Sym}^2 V \oplus \bigwedge^2 V$, $e_1: \text{Sym}^2 V \beta \rightarrow C \beta^* \rightarrow \bigwedge^2 (V)$ has non-trivial image. So, for example, the action of $B^3(0, -1)$ on $V \otimes 3$ does not restrict to a closed action on $(\text{Sym}^2 V) \otimes V$. 

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**Try 1:** For the partition $\lambda$ of size $\ell$, take the indecomposable $M(\lambda)$ indexed by $\lambda$ (the one paired with $B^\lambda$ by Moon, Kujawa-Tharp) in $V \otimes^\ell$. Write the action of $B_k(0, -1)$ on $M(\lambda) \otimes V \otimes^k$ in terms of the the action of $B_k(0, -1)$ on $V \otimes^{\ell+k}$; make an inductive argument.
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Let $\phi = \sum_{\alpha \in \Delta(g_1)} \alpha$ and let $V(\lambda)$ be the simple $g_0$-module of highest weight $\lambda$. Define

$$K(\lambda) = \text{Ind}_{g_0 \oplus g_1}^g V(\lambda - \phi) \quad \tilde{K}(\lambda) = \text{Ind}_{g_0 \oplus g_1}^g V(\lambda).$$
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Then $K(\lambda) \otimes V \cong M_1 \oplus \cdots \oplus M_n$ where

$$0 \rightarrow K(\lambda + \varepsilon_i) \rightarrow M_i \rightarrow K(\lambda - \varepsilon_i) \rightarrow 0,$$

whenever $\lambda \pm \varepsilon_i$ are dominant, or replace $K(*)$ with 0 whenever they’re not (similar statement for $\tilde{K}$).
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