Representation theory and combinatorics of diagram algebras.

Zajj Daugherty

May 15, 2016
Combinatorial representation theory

- What are the $A$-modules/representations?
- What are the simple/indecomposable $A$-modules/reps?
- What is the action of the center of $A$?
- What are their dimensions?
- How can I combine modules to make new ones, and what are they in terms of the simple modules?

In combinatorial representation theory, we use combinatorial objects to index (construct a bijection to) modules and representations, and to encode information about them.
Combinatorial representation theory

Representation theory: Given an algebra $A$...

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In combinatorial representation theory, we use combinatorial objects to index (construct a bijection to) modules and representations, and to encode information about them.
Motivating example: Schur-Weyl Duality

The **symmetric group** $S_k$ (permutations) as diagrams:
Motivating example: Schur-Weyl Duality

The symmetric group $S_k$ (permutations) as diagrams:

(with multiplication given by concatenation)
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$GL_n(\mathbb{C})$ acts on $\mathbb{C}^n \otimes \mathbb{C}^n \otimes \cdots \otimes \mathbb{C}^n = (\mathbb{C}^n)^\otimes_k$ diagonally.

$$g \cdot (v_1 \otimes v_2 \otimes \cdots \otimes v_k) = gv_1 \otimes gv_2 \otimes \cdots \otimes gv_k.$$
Motivating example: Schur-Weyl Duality

\[ GL_n(\mathbb{C}) \text{ acts on } \mathbb{C}^n \otimes \mathbb{C}^n \otimes \cdots \otimes \mathbb{C}^n = (\mathbb{C}^n)^\otimes k \text{ diagonally.} \]

\[ g \cdot (v_1 \otimes v_2 \otimes \cdots \otimes v_k) = gv_1 \otimes gv_2 \otimes \cdots \otimes gv_k. \]

\( S_k \) also acts on \((\mathbb{C}^n)^\otimes k\) by place permutation.
Motivating example: Schur-Weyl Duality

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$$g \cdot (v_1 \otimes v_2 \otimes \cdots \otimes v_k) = gv_1 \otimes gv_2 \otimes \cdots \otimes gv_k.$$ 

$S_k$ also acts on $(\mathbb{C}^n)^\otimes k$ by place permutation.

These actions commute!
Motivating example: Schur-Weyl Duality

Schur (1901): $S_k$ and $GL_n$ have commuting actions on $(\mathbb{C}^n)^\otimes k$.

Even better,

\[
\text{End}_{GL_n}((\mathbb{C}^n)^\otimes k) = \pi(\mathbb{C}S_k) \quad \text{and} \quad \text{End}_{S_k}((\mathbb{C}^n)^\otimes k) = \rho(\mathbb{C}GL_n).
\]

(all linear maps that commute with $GL_n$) (img of $S_k$ action) (img of $GL_n$ action)
Motivating example: Schur-Weyl Duality

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$$

(all linear maps that commute with $GL_n$) (img of $S_k$ action) (img of $GL_n$ action)

Why this is exciting:

The double-centralizer relationship produces

$$
(\mathbb{C}^n)^\otimes k \cong \bigoplus_{\lambda \vdash k} G^\lambda \otimes S^\lambda \quad \text{as a } GL_n-S_k \text{ bimodule,}
$$

where $G^\lambda$ are distinct irreducible $GL_n$-modules

$S^\lambda$ are distinct irreducible $S_k$-modules
Motivating example: Schur-Weyl Duality

Schur (1901): $S_k$ and $GL_n$ have commuting actions on $(\mathbb{C}^n)^\otimes k$.

Even better,

\[
\text{End}_{GL_n} ( (\mathbb{C}^n)^\otimes k ) = \pi ( \mathbb{C} S_k ) \quad \text{and} \quad \text{End}_{S_k} ( (\mathbb{C}^n)^\otimes k ) = \rho ( \mathbb{C} GL_n ).
\]

(End of linear maps that commute with $GL_n$) \quad (Img of $S_k$ action)

Why this is exciting:

The double-centralizer relationship produces

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(\mathbb{C}^n)^\otimes k \cong \bigoplus_{\lambda \vdash k} G^\lambda \otimes S^\lambda \quad \text{as a $GL_n$-$S_k$ bimodule},
\]

where $G^\lambda$ are distinct irreducible $GL_n$-modules

$S^\lambda$ are distinct irreducible $S_k$-modules

For example,

\[
\mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n = \bigoplus \left( G^{\lambda_1} \otimes S^{\lambda_2} \right) \]

\[
\bigoplus \left( G^{\lambda_3} \otimes S^{\lambda_4} \right) \bigoplus \left( G^{\lambda_5} \otimes S^{\lambda_6} \right)
\]
Representation theory of $V^\otimes k$

$V = \mathbb{C} = L(\square)$
Representation theory of $V^\otimes k$

$V = \mathbb{C} = L(\square), \quad L(\square)$

∅

\[
\begin{array}{c}
\emptyset \\
\downarrow \\
\square
\end{array}
\]
Representation theory of $V^\otimes k$

$V = \mathbb{C} = L(\square)$, \quad $L(\square) \otimes L(\square)$

∅

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|   |   |
Representation theory of $V^\otimes k$

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Representation theory of $V^\otimes k$

$$V = \mathbb{C} = L(\square), \quad L(\square) \otimes L(\square) \otimes L(\square) \otimes L(\square)$$

\[
\begin{array}{c}
\emptyset \\
\quad \\
\end{array}
\]
Representation theory of $V^\otimes k$

$$V = \mathbb{C} = L(\square), \quad L(\square) \otimes L(\square) \otimes L(\square) \otimes L(\square) \otimes L(\square) \cdots$$
Representation theory of $V^\otimes k$

\[ V = \mathbb{C} = L(\square), \quad L(\square) \otimes L(\square) \otimes L(\square) \otimes L(\square) \otimes \cdots \]

\[ \emptyset \]

\[ \cdots \]

\[ \cdots \]
Representation theory of $V^\otimes k$

\[ V = \mathbb{C} = L(\square), \quad L(\square) \otimes L(\square) \otimes L(\square) \otimes L(\square) \otimes L(\square) \cdots \]

\[
\begin{array}{c}
\emptyset \\
\downarrow \\
\vdots \\
\end{array}
\]
More centralizer algebras

Brauer (1937)
Orthogonal and symplectic groups
(and Lie algebras) acting on
$(\mathbb{C}^n)^\otimes k$ diagonally centralize
the Brauer algebra:

\[
\sum_{i=1}^{n} \delta_{b,c} v_i \otimes v_i \otimes v_a \otimes v_d \otimes v_d
\]

with $\bigcirc = n$
More centralizer algebras

Rep theory of $V \otimes^k$, orthogonal and symplectic:

$$V = \mathbb{C} = L(\square)$$
More centralizer algebras

Rep theory of $V^\otimes_k$, orthogonal and symplectic:

$V = \mathbb{C} = L(\square), \quad L(\square)$
More centralizer algebras

Rep theory of $V \otimes^k$, orthogonal and symplectic:

$V = \mathbb{C} = L(\Box), \quad L(\Box) \otimes L(\Box)$

\[
\begin{array}{c}
\emptyset \\
\downarrow \\
\Box \\
\downarrow \\
\emptyset \\
\end{array}
\quad
\begin{array}{c}
\emptyset \\
\downarrow \\
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\quad
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Rep theory of $V \otimes^k$, orthogonal and symplectic:

$V = \mathbb{C} = L(\square), \quad L(\square) \otimes L(\square) \otimes L(\square) \otimes L(\square) \cdots$
More centralizer algebras

Brauer (1937)
Orthogonal and symplectic groups (and Lie algebras) acting on $(\mathbb{C}^n)^\otimes k$ diagonally centralize the Brauer algebra:

$$\delta_{b,c} \sum_{i=1}^{n} v_i \otimes v_i \otimes v_a \otimes v_d \otimes v_d$$

with $\bigcirc = n$

Temperley-Lieb (1971)
GL$_2$ and SL$_2$ (and gl$_2$ and sl$_2$) acting on $(\mathbb{C}^2)^\otimes k$ diagonally centralize the Temperley-Lieb algebra:

$$\delta_{c,d} \sum_{i=1}^{2} v_a \otimes v_i \otimes v_i \otimes v_b \otimes v_e$$

with $\bigcirc = 2$
More centralizer algebras

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Orthogonal and symplectic groups (and Lie algebras) acting on $(\mathbb{C}^n)^{\otimes k}$ diagonally centralize the Brauer algebra:

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$$\delta_{c,d} \sum_{i=1}^{2} v_a \otimes v_i \otimes v_i \otimes v_b \otimes v_e$$

with $\bigcirc = 2$

Either way:
Diagrams encoding maps $V^{\otimes k} \rightarrow V^{\otimes k}$ that commute with the action of some classical algebra.
Quantum groups and braids

Fix $q \in \mathbb{C}$, and let $\mathcal{U} = \mathcal{U}_q \mathfrak{g}$ be the Drinfeld-Jimbo quantum group associated to Lie algebra $\mathfrak{g}$. 
Quantum groups and braids

Fix $q \in \mathbb{C}$, and let $\mathcal{U} = \mathcal{U}_q \mathfrak{g}$ be the Drinfeld-Jimbo quantum group associated to Lie algebra $\mathfrak{g}$.

$\mathcal{U} \otimes \mathcal{U}$ has an invertible element $\mathcal{R} = \sum \mathcal{R} R_1 \otimes R_2$ that yields a map

$$
\tilde{\mathcal{R}}_{VW} : V \otimes W \longrightarrow W \otimes V
$$

that

1. satisfies braid relations, and
2. commutes with the action on $V \otimes W$

for any $\mathcal{U}$-module $V$. 

\[\begin{array}{c}
\text{W} \otimes V \\
\text{V} \otimes W
\end{array}\]
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The braid group shares a commuting action with $\mathcal{U}$ on $V^\otimes_k$:
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\[ \tilde{R}_{VW} : V \otimes W \rightarrow W \otimes V \]

that (1) satisfies braid relations, and (2) commutes with the action on $V \otimes W$ for any $\mathcal{U}$-module $V$.

The one-pole/affine braid group shares a commuting action with $\mathcal{U}$ on $M \otimes V^\otimes k$:

\[ \tilde{R}_{MV} \tilde{R}_{VM} = \tilde{R}_{MV} \tilde{R}_{VM} \]

Around the pole:
Quantum groups and braids

Fix $q \in \mathbb{C}$, and let $\mathcal{U} = \mathcal{U}_q \mathfrak{g}$ be the Drinfeld-Jimbo quantum group associated to Lie algebra $\mathfrak{g}$.

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that

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The two-pole braid group shares a commuting action with $\mathcal{U}$ on $M \otimes V \otimes k \otimes N$:

Around the pole:

$$= \tilde{R}_{MV} \tilde{R}_{VM}$$
Type B, C, D
(orthog. & sympl.)

Type A
(gen. & sp. linear)

Small Type A
(GL$_2$ & SL$_2$)

Brauer algebra

Sym. group

Temperley-Lieb

Lie grp/alg
Brauer algebra
Sym. group
Temperley-Lieb

$V = \square$

$\Lambda \otimes \cdots \otimes \Lambda$

One-boundary TL

Two-boundary TL

$M \check\rightarrow \downarrow \check\rightarrow V \check\rightarrow k \check\rightarrow \check\rightarrow N$
Universal Type B, C, D
(orthog. & sympl.)

Type A
(gen. & sp. linear)

Small Type A
(GL₂ & SL₂)

Lie grp/alg

Brauer algebra

Sym. group

Temperley-Lieb

Braid group

BMW algebra

Hecke algebra

V = □

V ⊗ Λ

Lambda
Universal

Type B, C, D
(orthog. & sympl.)

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Hecke algebra

\[ \dim V = \sum \dim A \]

Affine braids

Affine BMW

Affine Hecke of type A (+twists)

One-boundary TL

Two-pole braids

Two-pole BMW

Affine Hecke of type C (+twists)

Two-boundary TL
Universal
(orthog. & sympl.)
(Brauer algebra)
(Braid group)
(Affine braids)
(Two-pole braids)

Type B, C, D
(GL$^2$ & $SL^2$)
(Sym. group)
(BMW algebra)
(Affine BMW)
(Two-pole BMW)

Type A
(gen. & sp. linear)
(Hecke algebra)
(Affine Hecke of type A (+twists))

Small Type A
 أثن. Hecke of type C (+twists)

Lie grp/alg
(Brauer algebra)
(Sym. group)
(Temperley-Lieb)

Quantum groups
(Affine braids)
(Affine BMW)
(Affine Hecke of type A (+twists))
(One-boundary TL)

Two-boundary TL

$V = \square$
$\Lambda \otimes \otimes \Lambda$
$M \otimes (\otimes \otimes N)$
$M(\otimes \otimes N) \otimes$
Type B, C, D
(orthog. & sympl.)

- Brauer algebra
- BMW algebra
- Affine BMW
- Two-pole BMW

Quantum groups

Nazarov (95): degenerate affine BMW algebras

H"aring-Oldenburg (98) and Orellana-Ram (04): affine BMW algebras act on $M \otimes V \otimes k$, commuting with the action of the quantum groups of types B, C, D.

Centralizer perspective: (D.-Ram-Virk) Use centralizer relationships to study these the affine and degenerate affine algebras simultaneously (representation theory of the quantum groups and the Lie algebras are basically the same).

Some results:
(a) The center of each algebra.
(b) Difficult "admissibility conditions" handled.
(c) Powerful "intertwiner" operators.
Nazarov (95): degenerate affine BMW algebras

\[ \ell \begin{array}{c} \textcircled{a} \\ \textcircled{b} \end{array} = z_\ell \in \mathbb{C} \]

act on \( M \otimes V \otimes k \), commuting with the action of the Lie algebras of types B, C, D.

**Type B, C, D**
(orthog. & sympl.)

**Lie grp/alg**

- **Brauer algebra**
- **BMW algebra**
- **Affine BMW**
- **Two-pole BMW**

**Quantum groups**

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Universal Type B, C, D (orthog. & sympl.)

Type A (gen. & sp. linear)

Small Type A (GL₂ & SL₂)

Lie grp/alg

Brauer algebra

Sym. group

Temperley-Lieb

Braid group

BMW algebra

Hecke algebra

Affine braids

Affine BMW

Affine Hecke of type A (+twists)

One-boundary TL

Two-pole braids

Two-pole BMW

Affine Hecke of type C (+twists)

Two-boundary TL

Quantum groups

= V ⊗ \cdots ⊗ V

= \Lambda \otimes \cdots \otimes \Lambda

M(\otimes \Lambda) \otimes M(\otimes \Lambda)

M(\otimes \Lambda) \otimes N

M(\otimes \Lambda) \otimes M(\otimes \Lambda)
Universal Type B, C, D (orthog. & sympl.)
Type A (gen. & sp. linear)
Small Type A (GL₂ & SL₂)

Lie grp/alg
Brauer algebra
Sym. group
Temperley-Lieb

Braid group
BMW algebra
Hecke algebra
Affine braids
Affine BMW
Affine Hecke
Affine Hecke of type A
Affine Hecke of type C

Quantum groups
One-boundary TL
Two-boundary TL

Two-pole braids
Two-pole BMW
Affine Hecke of type C (+twists)
Two boundary algebras:
Mitra, Nienhuis, De Gier, Batchelor (2004): Studying the six-vertex model with additional integrable boundary terms, introduced the two-boundary Temperley-Lieb algebra $TL_k$: even dots correspond to non-crossing diagrams.
De Gier, Nichols (2008): Explored representation theory of $TL_k$ using diagrams and established a connection to the affine Hecke algebras of type A and C.
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De Gier, Nichols (2008): Explored representation theory of $TL_k$ using diagrams and established a connection to the affine Hecke algebras of type A and C.
Affine type C Hecke algebra and two-boundary braids

Fix constants \( t_0, t_k, \) and \( t = t_1 = \cdots = t_{k-1} \). The affine Hecke algebra of type C, \( \mathcal{H}_k \), is generated by \( T_0, T_1, \ldots, T_k \) with relations

\[
T_i T_j \ldots = T_j T_i \ldots \quad \text{where} \quad m_{i,j} = \begin{cases} 
2 & \text{if } i = j \\
3 & \text{if } \left| i - j \right| = 1 \\
4 & \text{if } \left| i - j \right| = 2 
\end{cases}
\]

and \( T_i^2 = (t_i^{1/2} - t_i^{-1/2})T_i + 1 \).
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The two-boundary (two-pole) braid group $B_k$ is generated by

$$T_k = \begin{array}{c}
\bullet \\
\downarrow \\
\end{array}, \quad T_0 = \begin{array}{c}
\bullet \\
\uparrow \\
\end{array} \quad \text{and} \quad T_i = \begin{array}{c}
\bullet \\
\downarrow \\
\end{array} \quad \text{for } 1 \leq i \leq k - 1.$$
Fix constants $t_0, t_k$, and $t = t_1 = \cdots = t_{k-1}$. The affine Hecke algebra of type C, $H_k$, is generated by $T_0, T_1, \ldots, T_k$ with relations

$$T_i T_j \ldots = T_j T_i \ldots$$

and

$$T_i^2 = (t_i^{1/2} - t_i^{-1/2}) T_i + 1.$$

The two-boundary (two-pole) braid group $B_k$ is generated by

$$T_k = \includegraphics[scale=0.5]{diagram1}, \quad T_0 = \includegraphics[scale=0.5]{diagram2} \quad \text{and} \quad T_i = \includegraphics[scale=0.5]{diagram3} \quad \text{for} \ 1 \leq i \leq k - 1.$$

Relations:

$$T_i T_{i+1} T_i = \includegraphics[scale=0.5]{diagram4} = \includegraphics[scale=0.5]{diagram5} = T_{i+1} T_i T_{i+1}$$
Affine type C Hecke algebra and two-boundary braids

Fix constants $t_0, t_k$, and $t = t_1 = \cdots = t_{k-1}$. The affine Hecke algebra of type C, $\mathcal{H}_k$, is generated by $T_0, T_1, \ldots, T_k$ with relations

$$T_i T_j \ldots = T_j T_i \ldots$$

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The two-boundary (two-pole) braid group $B_k$ is generated by

$$T_k = \quad T_0 = \quad \text{and} \quad T_i = \quad \text{for } 1 \leq i \leq k - 1.$$

Relations:

$$T_i T_{i+1} T_i = \quad = \quad = T_{i+1} T_i T_{i+1}$$

$$T_1 T_0 T_1 T_0 = \quad = \quad = T_0 T_1 T_0 T_1$$
Affine type C Hecke algebra and two-boundary braids

Punchline:

- For any complex reductive Lie algebras \( \mathfrak{g} \), the quantum group \( \mathcal{U}_q \mathfrak{g} \) and the two-boundary braid group \( B_k \) have commuting actions on \( M \otimes (V)^\otimes k \otimes N \).

- When \( \mathfrak{g} = \mathfrak{gl}_n \), for good choices of \( M, N, \) and \( V \), the action of the two-boundary braid group factors to an action of the affine Hecke algebra of type \( C \).

Some consequences:

(a) A combinatorial classification and construction of irreducible representations of \( H_k \) (type C with distinct parameters).

(b) A diagrammatic intuition for \( H_k \).

(c) A classification of the representations of \( TL_k \) via the action of its center.
Affine type C Hecke algebra and two-boundary braids

Punchline:

- For any complex reductive Lie algebras $\mathfrak{g}$, the quantum group $\mathcal{U}_q\mathfrak{g}$ and the two-boundary braid group $B_k$ have commuting actions on $M \otimes (V)^\otimes k \otimes N$.

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Affine braids

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Affine Hecke of type A (+twists)

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Two-pole braids

Two-pole BMW

Affine Hecke of type C (+twists)

Two-boundary TL

Lie grp/alg

Quantum groups

\[ V = \bigotimes \cdots \bigotimes \Lambda \]

\[ (V \otimes \mathfrak{sl}_k)(V \otimes \mathfrak{sl}_k) \]