Representation theory and combinatorics of tensor power centralizer algebras

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(Joint work in progress with Arun Ram)

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Motivating example: Schur-Weyl Duality

The **symmetric group** $S_k$ (permutations) as diagrams:
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The **symmetric group** $S_k$ (permutations) as diagrams:

(with multiplication given by concatenation)
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Motivating example: Schur-Weyl Duality

$$\text{GL}_n(\mathbb{C}) \text{ acts on } \mathbb{C}^n \otimes \mathbb{C}^n \otimes \cdots \otimes \mathbb{C}^n = (\mathbb{C}^n)^\otimes k \text{ diagonally.}$$

$$g \cdot (v_1 \otimes v_2 \otimes \cdots \otimes v_k) = gv_1 \otimes gv_2 \otimes \cdots \otimes gv_k.$$
Motivating example: Schur-Weyl Duality

$\text{GL}_n(\mathbb{C})$ acts on $\mathbb{C}^n \otimes \mathbb{C}^n \otimes \cdots \otimes \mathbb{C}^n = (\mathbb{C}^n)^{\otimes k}$ diagonally.

$$g \cdot (v_1 \otimes v_2 \otimes \cdots \otimes v_k) = gv_1 \otimes gv_2 \otimes \cdots \otimes gv_k.$$ 

$S_k$ also acts on $(\mathbb{C}^n)^{\otimes k}$ by place permutation.
Motivating example: Schur-Weyl Duality

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\[ g \cdot (v_1 \otimes v_2 \otimes \cdots \otimes v_k) = gv_1 \otimes gv_2 \otimes \cdots \otimes gv_k. \]

\( S_k \) also acts on \( (\mathbb{C}^n)^{\otimes k} \) by place permutation.

These actions commute!
Motivating example: Schur-Weyl Duality

Schur (1901):

\[ \text{End}_{GL_n} \left( (\mathbb{C}^n \otimes k) \right) = \pi(\mathbb{C}S_k) \quad \text{and} \quad \text{End}_{S_k} \left( (\mathbb{C}^n \otimes k) \right) = \rho(\mathbb{C}GL_n). \]

(all linear maps that commute with $GL_n$) (img of $S_k$ action)

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Motivating example: Schur-Weyl Duality

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\]

(all linear maps that commute with \(GL_n\))

(img of \(S_k\) action)

(img of \(GL_n\) action)

Powerful consequence: a duality between representations

The double-centralizer relationship produces

\[
(\mathbb{C}^n) \otimes k \cong \bigoplus_{\lambda \vdash k} G^\lambda \otimes S^\lambda
\]

as a \(GL_n-S_k\) bimodule,

where \(G^\lambda\) are distinct irreducible \(GL_n\)-modules

\(S^\lambda\) are distinct irreducible \(S_k\)-modules
More centralizer algebras

Brauer (1937)
Orthogonal and symplectic groups
(and Lie algebras) acting on
\((\mathbb{C}^n)^{\otimes k}\) diagonally centralize
the **Brauer algebra**:

\[
\delta_{b,c} \sum_{i=1}^{n} v_i \otimes v_i \otimes v_a \otimes v_d \otimes v_d
\]

with \(\bigcirc = n\)

(Diagrams encode maps \(V^{\otimes k} \to V^{\otimes k}\) that commute with the action of some classical algebra.)
More centralizer algebras

Brauer (1937)
Orthogonal and symplectic groups (and Lie algebras) acting on $(\mathbb{C}^n)^\otimes k$ diagonally centralize the Brauer algebra:

\[ \delta_{b,c} \sum_{i=1}^{n} v_i \otimes v_i \otimes v_a \otimes v_d \otimes v_d \]

with \( \square = n \)

Temperley-Lieb (1971)
GL\(_2\) and SL\(_2\) (and \(\mathfrak{gl}_2\) and \(\mathfrak{sl}_2\)) acting on \((\mathbb{C}^2)^\otimes k\) diagonally centralize the Temperley-Lieb algebra:

\[ \delta_{c,d} \sum_{i=1}^{2} v_a \otimes v_i \otimes v_i \otimes v_b \otimes v_e \]

with \( \square = 2 \)

(Diagrams encode maps \( V^\otimes k \to V^\otimes k \) that commute with the action of some classical algebra.)
Quantum groups and braids

Fix $q \in \mathbb{C}$, and let $\mathcal{U} = \mathcal{U}_q g$ be the Drinfeld-Jimbo quantum group associated to Lie algebra $g$. 
Quantum groups and braids

Fix $q \in \mathbb{C}$, and let $\mathcal{U} = \mathcal{U}_q \mathfrak{g}$ be the Drinfeld-Jimbo quantum group associated to Lie algebra $\mathfrak{g}$.

$\mathcal{U} \otimes \mathcal{U}$ has an invertible element $R = \sum_R R_1 \otimes R_2$ that yields a map

$$\tilde{\mathcal{R}}_{VW} : V \otimes W \longrightarrow W \otimes V$$

that (1) satisfies braid relations, and (2) commutes with the $\mathcal{U}$-action on $V \otimes W$ for any $\mathcal{U}$-module $V$. 
Quantum groups and braids

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The braid group shares a commuting action with $\mathcal{U}$ on $V^\otimes k$:
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(1) satisfies braid relations, and 
(2) commutes with the $\mathcal{U}$-action on $V \otimes W$

for any $\mathcal{U}$-module $V$.

The one-pole/affine braid group shares a commuting action with $\mathcal{U}$ on $M \otimes V^\otimes k$:

Around the pole:

$$\mathcal{R}_{MV} \mathcal{R}_{VM}$$
Quantum groups and braids

Fix $q \in \mathbb{C}$, and let $\mathcal{U} = \mathcal{U}_q \mathfrak{g}$ be the Drinfeld-Jimbo quantum group associated to Lie algebra $\mathfrak{g}$.

$\mathcal{U} \otimes \mathcal{U}$ has an invertible element $\mathcal{R} = \sum_{\mathcal{R}} R_1 \otimes R_2$ that yields a map

$$\tilde{\mathcal{R}}_{VW} : V \otimes W \longrightarrow W \otimes V$$

that

- (1) satisfies braid relations, and
- (2) commutes with the $\mathcal{U}$-action on $V \otimes W$

for any $\mathcal{U}$-module $V$.

The two-pole braid group shares a commuting action with $\mathcal{U}$ on $M \otimes V^\otimes k \otimes N$:

Around the pole:

$$= \tilde{R}_{MV} \tilde{R}_{VM}$$
Universal
Type B, C, D
(orthog. & sympl.)
Type A
(gen. & sp. linear)
Small Type A
(GL₂ & SL₂)

Lie grp/alg
Brauer algebra
Sym. group
Temperley-Lieb

Braid group
BMW algebra
Hecke algebra

Affine braids
Affine BMW
Affine Hecke of type A (+twists)

Affine Hecke of type C (+twists)

Two-pole braids
Two-pole BMW

One-boundary TL
Two-boundary TL

Quantum groups
Two-boundary Temperley-Lieb algebras

[MNGB04] Fix $z, z_0, z_k \in \mathbb{C}$. The two-boundary Temperley-Lieb algebra $TL_k$ is a diagram algebra generated over $\mathbb{C}$ by diagrams

\[
e_0 = \begin{array}{c}
\includegraphics[scale=0.5]{e0.png}
\end{array}, \quad e_k = \begin{array}{c}
\includegraphics[scale=0.5]{ek.png}
\end{array}, \quad \text{and} \quad e_i = \begin{array}{c}
\includegraphics[scale=0.5]{ei.png}
\end{array}
\]

for $i = 1, \ldots, k - 1$
Two-boundary Temperley-Lieb algebras

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\[
e_0 = \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1
\end{array}
\end{array}
\end{array}
\end{array}
\end{array},
\quad
e_k = \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
k
\end{array}
\end{array}
\end{array}
\end{array}
\end{array},
\quad
\text{and}
\quad
e_i = \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
i
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\]

for $i = 1, \ldots, k - 1$, with relations $e_i e_j = e_j e_i$ for $|i - j| > 1,$

\[
e_i e_{i \pm 1} e_i = e_i
\]

for $1 \leq i \leq k - 1,$

\[
e_i^2 = z_i e_i.
\]
Two-boundary Temperley-Lieb algebras

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\[
e_0 = \begin{array}{c}
\begin{array}{c}
\includegraphics[width=2cm]{e0.png}
\end{array}
\end{array}, \quad
\begin{array}{c}
\begin{array}{c}
\includegraphics[width=2cm]{ek.png}
\end{array}
\end{array}, \quad \text{and} \quad
\begin{array}{c}
\begin{array}{c}
\includegraphics[width=2cm]{ei.png}
\end{array}
\end{array}
\]

for $i = 1, \ldots, k - 1$, with relations $e_ie_j = e_je_i$ for $|i - j| > 1$,

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Two-boundary Temperley-Lieb algebras

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for $i = 1, \ldots, k - 1$, with relations $e_i e_j = e_j e_i$ for $|i - j| > 1$,

$$ e_i e_{i \pm 1} e_i = e_i $$

for $1 \leq i \leq k - 1$,

$$ e_i^2 = z_i e_i. $$

or

or

or

or

or

or

or
Two-boundary Temperley-Lieb algebras

[MNGB04] Fix $z, z_0, z_k \in \mathbb{C}$. The two-boundary Temperley-Lieb algebra $TL_k$ is a diagram algebra generated over $\mathbb{C}$ by diagrams $e_0, e_k, e_i$ for $i = 1, \ldots, k - 1$, with relations $e_i e_j = e_j e_i$ for $|i - j| > 1$,

$e_i e_{i \pm 1} e_i = e_i$

for $1 \leq i \leq k - 1$,

$e_i^2 = z_i e_i$.

(Side loops are resolved with a 1 or a $z_i$ depending on whether there are an even or odd number of connections below their lowest point.)
Diagram multiplication:

In short, $\mathcal{TL}_k$ has basis given by non-crossing diagrams with

1. $k$ connections to the top and to the bottom,
2. an even number of connections to the right and to the left, and
3. no edges with both ends on the left or both ends on the right.

However, $2\ell \notin \mathcal{TL}_k$ so unlike the classical T-L algebras, $\mathcal{TL}_k$ is not finite dimensional!
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However, $\ell^2 \in \text{TL}_k$

So unlike the classical T-L algebras, $\text{TL}_k$ is not finite dimensional!
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\[
\begin{array}{c}
\text{Diagram multiplication:} \\
\end{array}
\]

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However, \( 2^\ell \in TL_k \)

So unlike the classical T-L algebras, \( TL_k \) is not finite dimensional!
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\[ \begin{array}{c}
\begin{array}{c}
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\end{array}
\end{array}\]

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\( \ast \) \( z \ast 1 \ast z_k \)
Diagram multiplication:

\[
\begin{array}{c}
\text{Diagram multiplication:} \\
\end{array}
\]

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3. no edges with both ends on the left or both ends on the right.

However,

\[
2\ell \in TL_k
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So unlike the classical T-L algebras, \( TL_k \) is not finite dimensional!
Universal

Type B, C, D
(orthog. & sympl.)

Type A
(gen. & sp. linear)

Small Type A
(GL$_2$ & SL$_2$)

Brauer algebra

Sym. group

Temperley-Lieb

Braid group

BMW algebra

Hecke algebra

Affine braids

Affine BMW

Affine Hecke
of type A
(+twists)

One-boundary TL

Affine Hecke
of type C
(+twists)

Two-boundary TL

Quantum groups

Lie grp/alg

V = \[\Lambda \otimes \ldots \otimes \Lambda\]

M \otimes (V \otimes N)

M \otimes (V \otimes N)

Two-pole braids

Two-pole BMW

Affine Hecke
of type C
(+twists)
Universal Type B, C, D
(orthog. & sympl.)
Type A
(gen. & sp. linear)
Small Type A
(GL₂ & SL₂)

Lie grp/alg
Brauer algebra
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Affine braids
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Affine Hecke of type A (+twists)

Two-pole braids
Two-pole BMW
Affine Hecke of type C (+twists)

One-boundary TL
Two-boundary TL
The two-boundary (two-pole) braid group $B_k$ is generated by

\[ T_k = \begin{array}{c}
\includegraphics{tikz/tikz_example1_tikz}
\end{array} , \quad T_0 = \begin{array}{c}
\includegraphics{tikz/tikz_example2_tikz}
\end{array} \quad \text{and} \quad T_i = \begin{array}{c}
\includegraphics{tikz/tikz_example3_tikz}
\end{array} \quad \text{for } 1 \leq i \leq k - 1, \]
The two-boundary (two-pole) braid group $\mathcal{B}_k$ is generated by

$$T_k = \begin{tikzpicture}[baseline=(current bounding box.center)]
\draw (0,0) -- (1,0);
\draw (0,0.5) -- (0,1);
\draw (0,0.5) .. controls ++(0.5,1) and ++(0.5,-1) .. (1,0);
\draw (0,1) .. controls ++(0.5,1) and ++(0.5,-1) .. (1,0);
\end{tikzpicture}, \quad T_0 = \begin{tikzpicture}[baseline=(current bounding box.center)]
\draw (0,0) -- (1,0);
\draw (0,0.5) -- (0,1);
\draw (0,0.5) .. controls ++(1,1) and ++(1,-1) .. (1,0);
\draw (0,1) .. controls ++(1,1) and ++(1,-1) .. (1,0);
\end{tikzpicture} \quad \text{and} \quad T_i = \begin{tikzpicture}[baseline=(current bounding box.center)]
\draw (0,0) -- (1,0);
\draw (0,0.5) -- (0,1);
\draw (0,0.5) .. controls ++(0.5,1) and ++(0.5,-1) .. (1,0);
\draw (0,1) .. controls ++(0.5,1) and ++(0.5,-1) .. (1,0);
\draw (0.5,0.5) -- (0.5,1);
\end{tikzpicture}$$

for $1 \leq i \leq k - 1$,

subject to relations

$$T_i T_{i+1} T_i = \begin{tikzpicture}[baseline=(current bounding box.center)]
\draw (0,0) -- (1,0);
\draw (0,0.5) -- (0,1);
\draw (0,0.5) .. controls ++(0.5,1) and ++(0.5,-1) .. (1,0);
\draw (0,1) .. controls ++(0.5,1) and ++(0.5,-1) .. (1,0);
\draw (0.5,0.5) -- (0.5,1);
\end{tikzpicture} = \begin{tikzpicture}[baseline=(current bounding box.center)]
\draw (0,0) -- (1,0);
\draw (0,0.5) -- (0,1);
\draw (0,0.5) .. controls ++(0.5,1) and ++(0.5,-1) .. (1,0);
\draw (0,1) .. controls ++(0.5,1) and ++(0.5,-1) .. (1,0);
\draw (0.5,0.5) -- (0.5,1);
\end{tikzpicture} = \begin{tikzpicture}[baseline=(current bounding box.center)]
\draw (0,0) -- (1,0);
\draw (0,0.5) -- (0,1);
\draw (0,0.5) .. controls ++(0.5,1) and ++(0.5,-1) .. (1,0);
\draw (0,1) .. controls ++(0.5,1) and ++(0.5,-1) .. (1,0);
\draw (0.5,0.5) -- (0.5,1);
\end{tikzpicture} = T_{i+1} T_i T_{i+1},$$
The two-boundary (two-pole) braid group $B_k$ is generated by

$$T_k = \begin{array}{c}
\begin{array}{c}
\text{\includegraphics[width=1cm]{tikz/tk.png}}
\end{array}
\end{array}, \quad T_0 = \begin{array}{c}
\begin{array}{c}
\text{\includegraphics[width=1cm]{tikz/t0.png}}
\end{array}
\end{array} \quad \text{and} \quad T_i = \begin{array}{c}
\begin{array}{c}
\text{\includegraphics[width=1cm]{tikz/ti.png}}
\end{array}
\end{array} \quad \text{for } 1 \leq i \leq k - 1,$$

subject to relations

$$T_i T_{i+1} T_i = \begin{array}{c}
\begin{array}{c}
\text{\includegraphics[width=1cm]{tikz/titi1.png}}
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\text{\includegraphics[width=1cm]{tikz/titi1.png}}
\end{array}
\end{array} = T_{i+1} T_i T_{i+1},$$

$$T_1 T_0 T_1 T_0 = \begin{array}{c}
\begin{array}{c}
\text{\includegraphics[width=1cm]{tikz/t1t0t1t0.png}}
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\text{\includegraphics[width=1cm]{tikz/t1t0t1t0.png}}
\end{array}
\end{array} = T_0 T_1 T_0 T_1,$$
The two-boundary (two-pole) braid group $\mathcal{B}_k$ is generated by

\[ T_k = \begin{array}{c}
\includegraphics{tikz/braid_k.png}
\end{array}, \quad T_0 = \begin{array}{c}
\includegraphics{tikz/braid_0.png}
\end{array} \text{ and } T_i = \begin{array}{c}
\includegraphics{tikz/braid_i.png}
\end{array} \text{ for } 1 \leq i \leq k - 1,
\]

subject to relations

\[ T_i T_{i+1} T_i = \begin{array}{c}
\includegraphics{tikz/braid_relations1.png}
\end{array} = \begin{array}{c}
\includegraphics{tikz/braid_relations2.png}
\end{array} = T_{i+1} T_i T_{i+1},
\]

\[ T_1 T_0 T_1 T_0 = \begin{array}{c}
\includegraphics{tikz/braid_relations3.png}
\end{array} = \begin{array}{c}
\includegraphics{tikz/braid_relations4.png}
\end{array} = T_0 T_1 T_0 T_1,
\]

and, similarly, \( T_{k-1} T_k T_{k-1} T_k = T_k T_{k-1} T_k T_{k-1} \).
The two-boundary (two-pole) braid group $\mathcal{B}_k$ is generated by

\[ T_k = \begin{array}{c}
\includegraphics[width=0.1\textwidth]{tikz/two-boundary.png}
\end{array}, \quad T_0 = \begin{array}{c}
\includegraphics[width=0.1\textwidth]{tikz/two-boundary.png}
\end{array} \quad \text{and} \quad T_i = \begin{array}{c}
\includegraphics[width=0.1\textwidth]{tikz/two-boundary.png}
\end{array} \quad \text{for } 1 \leq i \leq k - 1,
\]

subject to relations

\[ T_0 T_1 T_2 \cdots T_{k-2} T_{k-1} T_k \quad \text{i.e.}
\]

\[ T_i T_{i+1} T_i = \begin{array}{c}
\includegraphics[width=0.1\textwidth]{tikz/two-boundary.png}
\end{array} = \begin{array}{c}
\includegraphics[width=0.1\textwidth]{tikz/two-boundary.png}
\end{array} = T_{i+1} T_i T_{i+1},
\]

\[ T_1 T_0 T_1 T_0 = \begin{array}{c}
\includegraphics[width=0.1\textwidth]{tikz/two-boundary.png}
\end{array} = \begin{array}{c}
\includegraphics[width=0.1\textwidth]{tikz/two-boundary.png}
\end{array} = T_0 T_1 T_0 T_1,
\]

and, similarly, $T_{k-1} T_k T_{k-1} T_k = T_k T_{k-1} T_k T_{k-1}$. 
(1) The two-boundary (two-pole) braid group $B_k$ is generated by

$$T_k = \begin{array}{c}
\vdots \\
\bullet \\
\bullet \\
\bullet \\
\vdots \\
\end{array}, \quad T_0 = \begin{array}{c}
\vdots \\
\bullet \\
\bullet \\
\bullet \\
\vdots \\
\end{array} \quad \text{and} \quad T_i = \begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\vdots \\
\end{array} \quad \text{for } 1 \leq i \leq k - 1,$$

subject to relations

$$T_0 \quad T_1 \quad T_2 \quad T_{k-2} \quad T_{k-1} \quad T_k.$$

(2) Fix constants $t_0, t_k, t_i \in \mathbb{C}$.

The affine type C Hecke algebra $H_k$ is the quotient of $\mathbb{C}B_k$ by the relations

$$\left( T_0 - \frac{t_1}{2} \right) \left( T_0 + \frac{t-1}{2} \right) = 0,$$

$$\left( T_k - \frac{t_1}{2} \right) \left( T_k + \frac{t-1}{2} \right) = 0,$$

and

$$\left( T_i - \frac{t_1}{2} \right) \left( T_i + \frac{t-1}{2} \right) = 0$$

for $i = 1, \ldots, k - 1$.

(3) Set $e_j = t_1/2 - (e_0 = t_1/2 - T_0)$ and $e_k = t_1/2 - (e_k = t_1/2 - T_k)$ so that

$e_{2j} = z_j e_j$ (for good $z_j$).

The two-boundary Temperley-Lieb algebra is the quotient of $H_k$ by the relations

$e_i e_i \pm 1 e_i = e_i$ for $i = 1, \ldots, k - 1$. 
(1) The two-boundary (two-pole) braid group $B_k$ is generated by

$$T_k = \begin{array}{c}
\begin{array}{c}
\bigcirc
downarrow
\bigcirc
\end{array}
\end{array}, \quad T_0 = \begin{array}{c}
\begin{array}{c}
\bigcirc
downarrow
\bigcirc
\end{array}
\end{array} \text{ and } T_i = \begin{array}{c}
\begin{array}{c}
\bigcirc
downarrow
\bigcirc
\end{array}
\end{array} \quad \text{for } 1 \leq i \leq k - 1,$$

subject to relations

$$\begin{array}{c}
\begin{array}{c}
\bigcirc
downarrow
\bigcirc
\end{array}
\end{array} \bigcirc \begin{array}{c}
\begin{array}{c}
\bigcirc
downarrow
\bigcirc
\end{array}
\end{array} \bigcirc \begin{array}{c}
\begin{array}{c}
\bigcirc
downarrow
\bigcirc
\end{array}
\end{array} \bigcirc \begin{array}{c}
\begin{array}{c}
\bigcirc
downarrow
\bigcirc
\end{array}
\end{array} \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc.$$

(2) Fix constants $t_0, t_k, t \in \mathbb{C}$. The affine type C Hecke algebra $H_k$ is the quotient of $\mathbb{C}B_k$ by the relations

$$(T_0 - t_0^{1/2})(T_0 + t_0^{-1/2}) = 0, \quad (T_k - t_k^{1/2})(T_k + t_k^{-1/2}) = 0$$

and $$(T_i - t^{1/2})(T_i + t^{-1/2}) = 0 \quad \text{for } i = 1, \ldots, k - 1.$$
(1) The two-boundary (two-pole) braid group $B_k$ is generated by

$$
T_k = \begin{array}{c}
\begin{array}{c}
\text{\textbullet} \\
\text{\textbullet}
\end{array}
\end{array},
T_0 = \begin{array}{c}
\begin{array}{c}
\text{\textbullet} \\
\text{\textbullet}
\end{array}
\end{array} \quad \text{and} \quad T_i = \begin{array}{c}
\begin{array}{c}
\text{\textbullet} \\
\text{\textbullet}
\end{array}
\end{array} \quad \text{for } 1 \leq i \leq k - 1,
$$

subject to relations

$$
\begin{array}{c}
\begin{array}{c}
\text{\textbullet} \\
\text{\textbullet}
\end{array}
\end{array}\begin{array}{c}
\begin{array}{c}
\text{\textbullet} \\
\text{\textbullet}
\end{array}
\end{array}\begin{array}{c}
\begin{array}{c}
\text{\textbullet} \\
\text{\textbullet}
\end{array}
\end{array}\begin{array}{c}
\begin{array}{c}
\text{\textbullet} \\
\text{\textbullet}
\end{array}
\end{array}\begin{array}{c}
\begin{array}{c}
\text{\textbullet} \\
\text{\textbullet}
\end{array}
\end{array}.
\end{array}
$$

(2) Fix constants $t_0, t_k, t = t_1 = t_2 = \cdots = t_{k-1} \in \mathbb{C}$. The affine type C Hecke algebra $H_k$ is the quotient of $\mathbb{C}B_k$ by the relations $(T_i - t_i^{1/2})(T_i + t_i^{-1/2}) = 0$. 
(1) The two-boundary (two-pole) braid group $B_k$ is generated by

$$T_k = \begin{array}{c}
\begin{array}{c}
\circlearrowright \circlearrowleft
\end{array}
\end{array}, \quad T_0 = \begin{array}{c}
\begin{array}{c}
\circlearrowright \circlearrowleft
\end{array}
\end{array} \text{ and } T_i = \begin{array}{c}
\begin{array}{c}
\circlearrowright \circlearrowleft
\end{array}
\end{array} \text{ for } 1 \leq i \leq k - 1,$$

subject to relations

$$T_0 T_1 T_2 T_{k-2} T_{k-1} T_k.$$

(2) Fix constants $t_0, t_k, t = t_1 = t_2 = \cdots = t_{k-1} \in \mathbb{C}$. The affine type C Hecke algebra $\mathcal{H}_k$ is the quotient of $\mathbb{C}B_k$ by the relations $(T_i - t_i^{1/2})(T_i + t_i^{-1/2}) = 0$.

(3) Set

$$T_0 = t_0^{1/2} - \begin{array}{c}
\begin{array}{c}
\circlearrowright \circlearrowleft
\end{array}
\end{array}, \quad T_k = t_k^{1/2} - \begin{array}{c}
\begin{array}{c}
\circlearrowright \circlearrowleft
\end{array}
\end{array} \quad (e_0 = t_0^{1/2} - T_0)$$

$$T_k = t_k^{1/2} - \begin{array}{c}
\begin{array}{c}
\circlearrowright \circlearrowleft
\end{array}
\end{array} \quad (e_k = t_k^{1/2} - T_k)$$

$$T_i = t_i^{1/2} - \begin{array}{c}
\begin{array}{c}
\circlearrowright \circlearrowleft
\end{array}
\end{array} \quad (e_i = t_i^{1/2} - T_i)$$

so that $e_j^2 = z_j e_j$ (for good $z_j$).
(1) The two-boundary (two-pole) braid group $B_k$ is generated by

$$T_k = \begin{array}{c} \includegraphics[scale=0.5]{braid1} \end{array}, \quad T_0 = \begin{array}{c} \includegraphics[scale=0.5]{braid2} \end{array} \quad \text{and} \quad T_i = \begin{array}{c} \includegraphics[scale=0.5]{braid3} \end{array} \quad \text{for } 1 \leq i \leq k - 1,$$

subject to relations

$$T_0 T_1 T_2 T_{k-2} T_{k-1} T_k.$$

(2) Fix constants $t_0, t_k, t = t_1 = t_2 = \cdots = t_{k-1} \in \mathbb{C}$. The **affine type C Hecke algebra** $\mathcal{H}_k$ is the quotient of $\mathbb{C}B_k$ by the relations $(T_i - t_i^{1/2})(T_i + t_i^{-1/2}) = 0$.

(3) Set

$$e_0 = t_0^{1/2} - T_0 \quad \text{(for good } z_j \text{)}$$

$$e_k = t_k^{1/2} - T_k \quad \text{(for good } z_j \text{)}$$

$$e_i = t_i^{1/2} - T_i \quad \text{(for good } z_j \text{)}$$

so that $e_j^2 = z_j e_j$ (for good $z_j$).

The **two-boundary Temperley-Lieb algebra** is the quotient of $\mathcal{H}_k$ by the relations $e_i e_{i \pm 1} e_i = e_i$ for $i = 1, \ldots, k - 1$. 
(1) The two-boundary (two-pole) braid group $B_k$ is generated by

$$T_k = \begin{array}{c}
\includegraphics{example1.png}
\end{array}, \quad
T_0 = \begin{array}{c}
\includegraphics{example2.png}
\end{array} \quad \text{and} \quad
T_i = \begin{array}{c}
\includegraphics{example3.png}
\end{array} \quad \text{for } 1 \leq i \leq k - 1.$$

(2) Fix constants $t_0, t_k, t = t_1 = t_2 = \cdots = t_{k-1} \in \mathbb{C}$.

The affine type C Hecke algebra $\mathcal{H}_k$ is the quotient of $\mathbb{C}B_k$ by the relations $$(T_i - t_i^{1/2})(T_i + t_i^{-1/2}) = 0.$$ 

(3) Set

$$\begin{array}{c}
\includegraphics{example4.png}
\end{array} = t_0^{1/2} \begin{array}{c}
\includegraphics{example5.png}
\end{array} - \begin{array}{c}
\includegraphics{example6.png}
\end{array}, \quad
\begin{array}{c}
\includegraphics{example7.png}
\end{array} = t_k^{1/2} \begin{array}{c}
\includegraphics{example8.png}
\end{array} - \begin{array}{c}
\includegraphics{example9.png}
\end{array} \quad \text{and} \quad
\begin{array}{c}
\includegraphics{example10.png}
\end{array} = t^{1/2} \begin{array}{c}
\includegraphics{example11.png}
\end{array} - \begin{array}{c}
\includegraphics{example12.png}
\end{array}$$

so that $e_j^2 = z_j e_j$. The two-boundary Temperley-Lieb algebra is the quotient of $\mathcal{H}_k$ by the relations $e_i e_{i\pm 1} e_i = e_i$ for $i = 1, \ldots, k - 1$. 
(1) The two-boundary (two-pole) braid group $B_k$ is generated by

$$T_k = \begin{array}{c} \includegraphics[width=1cm]{tikz/braid}\end{array}, \quad T_0 = \begin{array}{c} \includegraphics[width=1cm]{tikz/braid}\end{array} \quad \text{and} \quad T_i = \begin{array}{c} \includegraphics[width=1cm]{tikz/braid}\end{array} \quad \text{for} \quad 1 \leq i \leq k - 1. \quad \text{(1)}$$

(2) Fix constants $t_0, t_k, t = t_1 = t_2 = \cdots = t_{k-1} \in \mathbb{C}$. The affine type C Hecke algebra $\mathcal{H}_k$ is the quotient of $\mathbb{C}B_k$ by the relations $(T_i - t_i^{1/2})(T_i + t_i^{-1/2}) = 0.$

(3) Set

$$\begin{array}{c} \includegraphics[width=1cm]{tikz/braid}\end{array} = t_0^{1/2} \begin{array}{c} \includegraphics[width=1cm]{tikz/braid}\end{array} - \begin{array}{c} \includegraphics[width=1cm]{tikz/braid}\end{array}, \quad \begin{array}{c} \includegraphics[width=1cm]{tikz/braid}\end{array} = t_k^{1/2} \begin{array}{c} \includegraphics[width=1cm]{tikz/braid}\end{array} - \begin{array}{c} \includegraphics[width=1cm]{tikz/braid}\end{array} \quad \text{and} \quad \begin{array}{c} \includegraphics[width=1cm]{tikz/braid}\end{array} = t^{1/2} \begin{array}{c} \includegraphics[width=1cm]{tikz/braid}\end{array} - \begin{array}{c} \includegraphics[width=1cm]{tikz/braid}\end{array} \end{array}$$

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### Universal

- **Type B, C, D**
- **Type A**
- **Small Type A**

<table>
<thead>
<tr>
<th><strong>Qu grp</strong></th>
<th>Universal</th>
<th>Type B, C, D</th>
<th>Type A</th>
<th>Small Type A</th>
</tr>
</thead>
<tbody>
<tr>
<td>n</td>
<td>Two-pole braids</td>
<td>Two-pole BMW</td>
<td>Affine Hecke of type C (+twists)</td>
<td>Two-boundary TL</td>
</tr>
</tbody>
</table>

$M \otimes V \otimes \mathbb{K}$
Representation theory of $\mathcal{H}_k$

The representations of $\mathcal{H}_k$ are indexed by pairs $(c, J)$, where

- $c$ is a point in the fundamental chamber of the (finite) type C hyperplane system, and
- $J$ is a set of choices of positive/negative sides of other distinguished hyperplanes intersecting $c$
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Example: $k = 2$
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The $r_i$s depend on $\mathcal{H}_k$'s parameters $t_0$ and $t_k$: $r_1 = \log_t (t_0 / t_k)$, $r_2 = \log_t (t_0 t_k)$
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