Representation theory of the two-boundary Temperley-Lieb algebra

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(Joint work in progress with Arun Ram)

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Temperley-Lieb algebras

The *Temperley-Lieb algebra* $TL_k(q)$ is the algebra of non-crossing pairings on $2k$ vertices

1 2 3 4 $k$

with multiplication given by stacking diagrams, subject to the relation

$$\bigcirc = q + q^{-1}$$
Temperley-Lieb algebras

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1  2  3  4  k

with multiplication given by stacking diagrams, subject to the relation

$$= q + q^{-1}$$

Multiplication:
Temperley-Lieb algebras

The *Temperley-Lieb algebra* $TL_k(q)$ is the algebra of non-crossing pairings on $2k$ vertices

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
\end{array}
\begin{array}{c}
\circ \circ \\
\end{array}
\begin{array}{c}
\circ \\
\end{array}
\begin{array}{c}
\circ \\
\end{array}
\begin{array}{c}
k \\
\end{array}
\]

with multiplication given by stacking diagrams, subject to the relation

\[
\bigcirc = q + q^{-1}
\]

Multiplication:
Temperley-Lieb algebras

The *Temperley-Lieb algebra* $\mathcal{TL}_k(q)$ is the algebra of non-crossing pairings on $2k$ vertices

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
2 & 3 & 4 & k \\
3 & 4 & k & 1 \\
4 & k & 1 & 2 \\
k & 1 & 2 & 3 \\
\end{array}
\]

with multiplication given by stacking diagrams, subject to the relation

\[
\bigcirc = q + q^{-1}
\]

**Multiplication:**

\[
\begin{array}{cccc}
\otimes & \otimes & \otimes & \otimes \\
\overset{\text{Multiplication}}{=} & \overset{\text{Multiplication}}{=} & \overset{\text{Multiplication}}{=} & \overset{\text{Multiplication}}{=} \\
\end{array}
\]
Temperley-Lieb algebras

The *Temperley-Lieb algebra* $TL_k(q)$ is the algebra of non-crossing pairings on $2k$ vertices

with multiplication given by stacking diagrams, subject to the relation

$\bigcirc = q + q^{-1}$

**Multiplication:**

\[\begin{array}{c}
\bigcirc \\
\bigcirc
\end{array} \quad = \quad \begin{array}{c}
\bigcirc \\
\bigcirc
\end{array} \quad \quad \quad \begin{array}{c}
\bigcirc \\
\bigcirc
\end{array} \quad * (q + q^{-1})^2\]
Temperley-Lieb algebras

The one-boundary Temperley-Lieb algebra $T L_k^{(1)}(q, z_0)$ is the algebra of one-walled non-crossing pairings on $2k$ vertices with multiplication given by stacking diagrams, subject to the relations

$$\bigcirc = q + q^{-1} \quad \text{and} \quad \text{if even } \# \text{ connections below} = 1 \quad \text{or} \quad \text{if odd } \# \text{ connections below} = z_0.$$
Odd/even relations

The algebra $T L_k^{(1)}(q, z_0)$ is generated by

$$
e_i = \begin{array}{c}
\text{\includegraphics[width=1.5cm]{diagram1.png}}
\end{array} 
\quad \text{and} \quad 
\begin{array}{c}
\text{\includegraphics[width=2.5cm]{diagram2.png}}
\end{array}$$

for $i = 1, \ldots, k - 1$
Odd/even relations

The algebra $TL_k^{(1)}(q, z_0)$ is generated by

$$e_i = \quad \text{and} \quad e_0 =$$

for $i = 1, \ldots, k - 1$, with relations

$$e_i e_{i \pm 1} e_i = e_i \quad \text{for} \quad i \geq 1$$
Odd/even relations

The algebra $TL^{(1)}_k(q, z_0)$ is generated by

$$e_i = \text{Diagram 1}$$
and
$$e_0 = \text{Diagram 2}$$

for $i = 1, \ldots, k - 1$, with relations

$$e_i e_{i+1} e_i = e_i \text{ for } i \geq 1$$

or

$$\text{Diagram 3} = \text{Diagram 4}$$
Odd/even relations

The algebra $TL^{(1)}_k(q, z_0)$ is generated by

$$e_i = \begin{array}{c}
\includegraphics[height=1cm]{diagram1}
\end{array} \text{ and } e_0 = \begin{array}{c}
\includegraphics[height=1cm]{diagram2}
\end{array}$$

for $i = 1, \ldots, k - 1$, with relations

$$e_i e_{i \pm 1} e_i = e_i \text{ for } i \geq 1$$

or

$$e_i^2 = ae_i$$

with $a = (q + q^{-1})$.
Odd/even relations

The algebra $TL_k^{(1)}(q, z_0)$ is generated by

$$e_i = \begin{array}{c}
\includegraphics{diagram1} \\
i
\end{array} \quad \text{and} \quad e_0 = \begin{array}{c}
\includegraphics{diagram2} \\
1
\end{array}$$

for $i = 1, \ldots, k - 1$, with relations

$$e_i e_{i \pm 1} e_i = e_i \quad \text{for } i \geq 1$$

$$e_i^2 = ae_i$$

Side loops are resolved with a $1$ or a $z_0$ depending on whether there are an even or odd number of connections below their lowest point.
Odd/even relations

The algebra $TL_k^{(1)}(q, z_0)$ is generated by

$$e_i = \begin{array}{c}
\includegraphics{figure1.png}
\end{array}$$

and

$$e_0 = \begin{array}{c}
\includegraphics{figure2.png}
\end{array}$$

for $i = 1, \ldots, k - 1$, with relations

$$e_i e_{i \pm 1} e_i = e_i \text{ for } i \geq 1$$

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Side loops are resolved with a 1 or a $z_0$ depending on whether there are an even or odd number of connections below their lowest point.
Temperley-Lieb algebras

The **one-boundary Temperley-Lieb algebra** $\mathcal{T}L_{k}^{(1)}(q, z_0)$ is the algebra of one-walled non-crossing pairings on $2k$ vertices

![Diagram](image)

with multiplication given by stacking diagrams, subject to the relations

$$
\bigcirc = q + q^{-1}
$$

and

- if even # connections below
- if odd # connections below

$$
= 1 \quad \text{or} \quad = z_0.
$$
Our main object: two-boundary Temperley-Lieb algebra

Nienhuis, De Gier, Batchelor (2004):

The two-boundary Temperley-Lieb algebra $\text{TL}_k^{(2)}(q, z_0, z_k) = \mathcal{T}_k$ is the algebra of two-walled non-crossing pairings on $2k$ vertices

so that each wall always has an even number of connections, with multiplication given by stacking diagrams, subject to the relations

$$\bigcirc = q + q^{-1}$$

and

if even # connections below

$$= 1$$

or

if odd # connections below

$$= z_0,$$

$$= z_k.$$
Our main object: two-boundary Temperley-Lieb algebra

Nienhuis, De Gier, Batchelor (2004):

The two-boundary Temperley-Lieb algebra $TL_k^{(2)}(q, z_0, z_k) = \mathcal{T}_k$ is the algebra of two-walled non-crossing pairings on $2k$ vertices

\[ \begin{array}{c}
1 \quad 2 \quad 3 \quad 4 \quad k \\
\bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \\
\end{array} \]

so that each wall always has an even number of connections, with multiplication given by stacking diagrams, subject to the relations

\[ \bigcirc = q + q^{-1} \quad \text{and} \]

\[ \begin{array}{c}
\begin{array}{c}
\text{if even #} \\
\text{connections below}
\end{array} \\
\begin{array}{c}
\text{if odd #} \\
\text{connections below}
\end{array} \\
\end{array} \]
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The two-boundary Temperley-Lieb algebra $TL_k^{(2)}(q, z_0, z_k) = \mathcal{T}_k$ is the algebra of two-walled non-crossing pairings on $2k$ vertices

so that each wall always has an even number of connections, with multiplication given by stacking diagrams, subject to the relations

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$$\text{if even } \# \text{ connections below} \quad = 1 \quad \text{or} \quad \text{if odd } \# \text{ connections below} \quad = z_0, \quad = z_k.$$
Our main object: two-boundary Temperley-Lieb algebra

$TL_k$ is finite-dimensional ($n$th Catalan number)
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$TL_k$ is finite-dimensional ($n$th Catalan number)

$TL_k^{(1)}$ is finite-dimensional
Our main object: two-boundary Temperley-Lieb algebra

\[ TL_k \] is finite-dimensional (\( n \)th Catalan number)

\[ TL_k^{(1)} \] is finite-dimensional

\[ TL_k^{(2)} = T_k \] is infinite-dimensional!
Our main object: two-boundary Temperley-Lieb algebra

\[ T L_k \] is finite-dimensional (\( n \)th Catalan number)

\[ T L_k^{(1)} \] is finite-dimensional

\[ T L_k^{(2)} = \mathcal{T}_k \] is infinite-dimensional!

de Gier, Nichols (2008): Explored representation theory of \( \mathcal{T}_k \).

1. Take quotients giving \( = z \) to get finite-dimensional algebras.

2. Establish connection to the affine Hecke algebras of type A and C to facilitate calculations.

3. Use diagrammatics and an action on \( (\mathbb{C}^2)^\otimes k \) to help classify representations in quotient (most modules are \( 2^k \) dim’l; some split).
Our main object: two-boundary Temperley-Lieb algebra

\[ TL_k \] is finite-dimensional (n\text{th} Catalan number) \quad \text{SWD✓}

\[ TL_k^{(1)} \] is finite-dimensional \quad \text{SWD✓}

\[ TL_k^{(2)} = T_k \] is infinite-dimensional!

\[ 2\ell \]

\begin{align*}
\text{de Gier, Nichols (2008): Explored representation theory of } T_k.
\end{align*}

1. Take quotients giving \[ = z \] to get finite-dimensional algebras.

2. Establish connection to the affine Hecke algebras of type A and C to facilitate calculations.

3. Use diagrammatics and an action on \((\mathbb{C}^2)^\otimes_k\) to help classify representations in quotient (most modules are \(2^k\) dim'l; some split).
Quantum groups and braids

Fix $q \in \mathbb{C}^*$. Let $U = U_q \mathfrak{g}$ be the Drinfel’d-Jimbo quantum group associated to a reductive Lie algebra $\mathfrak{g}$. Let $V, M$ be $U$-modules. Then $U \otimes U$ has invertible $R = \sum_R R_1 \otimes R_2$ that yields a map

$$\tilde{R}_{VM} : V \otimes M \rightarrow M \otimes V$$

$$v \otimes m \mapsto \sum_R R_1 m \otimes R_2 v$$

that

(1) satisfies braid relations, and
(2) commutes with the action of $U_q \mathfrak{g}$. 
Quantum groups and braids

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that (1) satisfies braid relations, and
(2) commutes with the action of $U_q \mathfrak{g}$.

The braid group shares a commuting action with $U_q \mathfrak{g}$ on $V^\otimes k$: 

![Diagram](image-url)
Quantum groups and braids

Fix $q \in \mathbb{C}^*$. Let $U = U_q g$ be the Drinfel’d-Jimbo quantum group associated to a reductive Lie algebra $g$. Let $V, M$ be $U$-modules. Then $U \otimes U$ has invertible $R = \sum R R_1 \otimes R_2$ that yields a map

$$\tilde{R}_{VM} : V \otimes M \longrightarrow M \otimes V$$

$$v \otimes m \longmapsto \sum R R_1 m \otimes R_2 v$$

that

1. satisfies braid relations, and
2. commutes with the action of $U_q g$.

The one-boundary/affine braid group shares a commuting action with $U_q g$ on $N \otimes V^\otimes k$:

Around the pole:

$$= \tilde{R}_{NV} \tilde{R}_{VN}$$
Quantum groups and braids

Fix $q \in \mathbb{C}^*$. Let $U = U_q \mathfrak{g}$ be the Drinfel’d-Jimbo quantum group associated to a reductive Lie algebra $\mathfrak{g}$. Let $V, M$ be $U$-modules. Then $U \otimes U$ has invertible $R = \sum_R R_1 \otimes R_2$ that yields a map

$$\tilde{R}_{VM} : V \otimes M \rightarrow M \otimes V$$

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that (1) satisfies braid relations, and
(2) commutes with the action of $U_q \mathfrak{g}$.

The two-boundary braid group shares a commuting action with $U_q \mathfrak{g}$ on $N \otimes V \otimes^k \otimes M$:

Around the pole:

$$= \tilde{R}_{NV} \tilde{R}_{VN}$$
Affine type C Hecke algebra and two-boundary braids

Fix constants $t_0, t_k$, and $t = t_1 = \cdots = t_{k-1}$. The affine Hecke algebra of type C, $\mathcal{H}_k$, is generated by $T_0, T_1, \ldots, T_k$ with relations

$$T_i T_j \cdots = T_j T_i \cdots$$

where $m_{i,j} =$

- 2 if
- 3 if
- 4 if

and $T_i^2 = (t_i^{1/2} - t_i^{-1/2})T_i + 1$. 

The two-boundary (two-pole) braid group $B_k$ is generated by $T_k = T_0 = T_1 = \cdots$ for $1 \leq i \leq k - 1$. Relations:

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$$

$$T_0 T_1 T_0 T_1 = T_0 T_1 T_0 T_1$$
Affine type C Hecke algebra and two-boundary braids

Fix constants \( t_0, t_k \), and \( t = t_1 = \cdots = t_{k-1} \). The affine Hecke algebra of type C, \( \mathcal{H}_k \), is generated by \( T_0, T_1, \ldots, T_k \) with relations

\[
T_i T_j \cdots = T_j T_i \cdots \quad \text{and} \quad T_i^2 = (t_i^{1/2} - t_i^{-1/2}) T_i + 1.
\]

\( m_{i,j} \) factors \( m_{i,j} \) factors
Affine type C Hecke algebra and two-boundary braids

Fix constants $t_0, t_k$, and $t = t_1 = \cdots = t_{k-1}$. The affine Hecke algebra of type C, $\mathcal{H}_k$, is generated by $T_0, T_1, \ldots, T_k$ with relations

$$T_i T_j \ldots = T_j T_i \ldots$$

and

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The two-boundary (two-pole) braid group $\mathcal{B}_k$ is generated by

$$T_k = \quad T_0 = \quad \text{and} \quad T_i = \quad \text{for } 1 \leq i \leq k - 1.$$
Affine type C Hecke algebra and two-boundary braids

Fix constants $t_0, t_k$, and $t = t_1 = \cdots = t_{k-1}$. The affine Hecke algebra of type C, $\mathcal{H}_k$, is generated by $T_0, T_1, \ldots, T_k$ with relations

$$T_i T_j \cdots = T_j T_i \cdots$$

and

$$T_i^2 = (t_i^{1/2} - t_i^{-1/2})T_i + 1.$$

The two-boundary (two-pole) braid group $\mathcal{B}_k$ is generated by

$$T_k = \begin{array}{c}
\includegraphics[width=1cm]{T_k.png}
\end{array} \quad T_0 = \begin{array}{c}
\includegraphics[width=1cm]{T_0.png}
\end{array} \quad \text{and} \quad T_i = \begin{array}{c}
\includegraphics[width=1cm]{T_i.png}
\end{array} \quad \text{for } 1 \leq i \leq k - 1.$$

Relations:

$$T_i T_{i+1} T_i = \begin{array}{c}
\includegraphics[width=2cm]{T_iT_i+1T_i.png}
\end{array} = \begin{array}{c}
\includegraphics[width=2cm]{T_iT_i.png}
\end{array} = T_{i+1} T_i T_{i+1}$$
Affine type C Hecke algebra and two-boundary braids

Fix constants $t_0, t_k$, and $t = t_1 = \cdots = t_{k-1}$. The affine Hecke algebra of type C, $\mathcal{H}_k$, is generated by $T_0, T_1, \ldots, T_k$ with relations

$$T_i T_j \cdots = T_j T_i \cdots$$

$\text{m}_{i,j}$ factors $\text{m}_{i,j}$ factors

and

$$T_i^2 = \left( t_i^{1/2} - t_i^{-1/2} \right) T_i + 1.$$

The two-boundary (two-pole) braid group $B_k$ is generated by

$$T_k = \quad T_0 = \quad \text{and} \quad T_i = \quad \text{for } 1 \leq i \leq k - 1.$$ 

Relations:

$$T_i T_{i+1} T_i = \quad = \quad = T_{i+1} T_i T_{i+1}$$

$$T_1 T_0 T_1 T_0 = \quad = \quad = T_0 T_1 T_0 T_1$$
Theorem (D.-Ram, degenerate versions of 1&2 in [D. 10])

(1) Let \( U = U_q \mathfrak{g} \) for any complex reductive Lie algebras \( \mathfrak{g} \).
Let \( M, N, \) and \( V \) be finite-dimensional modules.

The two-boundary braid group \( B_k \) acts on \( N \otimes (V)^k \otimes M \) and this action commutes with the action of \( U \).

(2) If \( \mathfrak{g} = \mathfrak{gl}_n \), then (for good simple choices of \( M, N, \) and \( V \)),
the affine Hecke algebra of type \( C, \mathcal{H}_k \), acts on \( N \otimes (V)^k \otimes M \)
and this action commutes with the action of \( U \).
Theorem (D.-Ram, degenerate versions of 1&2 in [D. 10])

(1) Let $U = U_q g$ for any complex reductive Lie algebras $g$. Let $M$, $N$, and $V$ be finite-dimensional modules. The two-boundary braid group $B_k$ acts on $N \otimes (V)^{\otimes k} \otimes M$ and this action commutes with the action of $U$.

(2) If $g = gl_n$, then (for good simple choices of $M$, $N$, and $V$), the affine Hecke algebra of type $C$, $H_k$, acts on $N \otimes (V)^{\otimes k} \otimes M$ and this action commutes with the action of $U$.

Now using braid diagrammatics, [GN 08] says that by identifying

$c_0 = t_0^{1/2}$, $c_k = t_k^{1/2}$

(where $c_i = t_i^{1/2} t_i^{-1/2} + t_i^{-1/2} t_i^{1/2}$),

then $\mathcal{T}_k$ is a quotient of $H_k$ by...
Theorem (D.-Ram, degenerate versions of 1&2 in [D. 10])

(1) Let $U = U_q \mathfrak{g}$ for any complex reductive Lie algebras $\mathfrak{g}$. Let $M$, $N$, and $V$ be finite-dimensional modules.

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(2) If $\mathfrak{g} = \mathfrak{gl}_n$, then (for good simple choices of $M$, $N$, and $V$), the affine Hecke algebra of type $C$, $\mathcal{H}_k$, acts on $N \otimes (V)^{\otimes k} \otimes M$ and this action commutes with the action of $U$.

Now using braid diagrammatics, [GN 08] says that by identifying

\[
\begin{align*}
\mathcal{T}_k & = t^{1/2} \left( - \left\langle \begin{array}{c}
\end{array} \right. \right) \\
\mathcal{T}_k & = t_0^{1/2} \left( - \left\langle \begin{array}{c}
\end{array} \right. \right) \\
\mathcal{T}_k & = t_k^{1/2} \left( - \left\langle \begin{array}{c}
\end{array} \right. \right)
\end{align*}
\]

(where $c_i = t_i^{1/2}t^{-1/2} + t_i^{-1/2}t^{1/2}$,

then $\mathcal{T}_k$ is a quotient of $\mathcal{H}_k$ by

\[
\begin{align*}
e_i e_{i \pm 1} e_i & \quad \text{for } 1 \leq i \leq k-1 : \\
\mathcal{T}_k & = e_i \\
\mathcal{T}_k & = 1 \\
\mathcal{T}_k & = 1 \left( \text{and reverses} \right)
\end{align*}
\]
Theorem (D.-Ram, degenerate versions of 1&2 in [D. 10])

(1) Let $U = U_q \mathfrak{g}$ for any complex reductive Lie algebras $\mathfrak{g}$. Let $M$, $N$, and $V$ be finite-dimensional modules.

The two-boundary braid group $B_k$ acts on $N \otimes (V)^{\otimes k} \otimes M$ and this action commutes with the action of $U$.

(2) If $\mathfrak{g} = \mathfrak{gl}_n$, then (for good simple choices of $M$, $N$, and $V$), the affine Hecke algebra of type $C$, $\mathcal{H}_k$, acts on $N \otimes (V)^{\otimes k} \otimes M$ and this action commutes with the action of $U$.

Now using braid diagrammatics, [GN 08] says that by identifying

\[\frac{c_i}{2} = t^{1/2} \quad \text{for } i = 1, \ldots, \frac{k}{2} \quad \text{and} \quad \frac{c_i}{2} = t^{-1/2} \quad \text{for } i = \frac{k}{2} + 1, \ldots, k,\]

then $T_k$ is a quotient of $\mathcal{H}_k$ by

\[e_i e_{i \pm 1} e_i \quad \text{for } 1 \leq i \leq k-1 : \quad \begin{array}{c} \includegraphics[width=0.5\textwidth]{diagram1} \\ \text{and} \end{array} \quad \begin{array}{c} \includegraphics[width=0.5\textwidth]{diagram2} \quad \text{reverses} \end{array}\]

(3) When $\mathfrak{g} = \mathfrak{gl}_2$, $T_k$ acts on $N \otimes (V)^{\otimes k} \otimes M$ (for good choices).
Consider the fin-dim’l simple $U_q\mathfrak{gl}_n$-modules $L(\lambda)$ indexed by partitions:

$$\lambda = \boxed{\quad \quad \quad \quad}$$

So let $M$ and $N$ be indexed by rectangular partitions, which have two addable boxes:

$$(a \quad c) = (c \quad a)$$

$H_k$ has a commuting action with $U_q\mathfrak{gl}_n$ on the space $L((b \quad d)) \otimes (L((a \quad c))) \otimes k \otimes L((a \quad c))$ with $c,d < n$. 
Consider the fin-dim’l simple $U_q\mathfrak{gl}_n$-modules $L(\lambda)$ indexed by partitions:

The content of a box is its diagonal number.
Consider the fin-dim’l simple $U_q \mathfrak{gl}_n$-modules $L(\lambda)$ indexed by partitions:

\[ \lambda = \begin{array}{cccc}
0 & 1 & 2 & 3 \\
-1 & & & \\
& & & \end{array} \]

The content of a box is its diagonal number.

Fix $V = L(\square)$. The generators of $\mathcal{H}_k$ acting on $N \otimes V^k \otimes M$ look like

\[ T_k = \begin{array}{c}
V \otimes M \\
V \otimes M 
\end{array} \]
\[ T_0 = \begin{array}{c}
N \otimes V \\
N \otimes V 
\end{array} \]
\[ T_i = \begin{array}{c}
V \otimes V \\
V \otimes V 
\end{array} \]

The eigenvalues of these operators (of which there should be two, since

\[(T_k - t_k^{1/2})(T_k + t_k^{-1/2}) = (T_0 - t_0^{1/2})(T_0 + t_0^{-1/2}) = (T_i - t^{1/2})(T_i + t^{-1/2}) = 0\]

are controlled by contents of addable boxes.
Consider the fin-dim’l simple $U_q \mathfrak{gl}_n$-modules $L(\lambda)$ indexed by partitions:

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are controlled by contents of addable boxes. So let $M$ and $N$ be indexed by rectangular partitions, which have two addable boxes:
Consider the fin-dim’l simple $U_q\mathfrak{gl}_n$-modules $L(\lambda)$ indexed by partitions:

The content of a box is its diagonal number. 

Fix $V = L(\square)$. The generators of $\mathcal{H}_k$ acting on $N \otimes V^\otimes k \otimes M$ look like

$$T_k = \begin{array}{c}
V \otimes M \\
\downarrow \\
V \otimes M
\end{array} \quad T_0 = \begin{array}{c}
N \otimes V \\
\downarrow \\
N \otimes V
\end{array} \quad \text{and} \quad T_i = \begin{array}{c}
V \otimes V \\
\downarrow \\
V \otimes V
\end{array}$$

The eigenvalues of these operators (of which there should be two, since

$$(T_k - t_k^{1/2})(T_k + t_k^{-1/2}) = (T_0 - t_0^{1/2})(T_0 + t_0^{-1/2}) = (T_i - t^{1/2})(T_i + t^{-1/2}) = 0$$

are controlled by contents of addable boxes. So let $M$ and $N$ be indexed by rectangular partitions, which have two addable boxes:

$$\begin{array}{c}
\begin{array}{c}
N \otimes V \\
\downarrow \\
N \otimes V
\end{array}
\end{array} \quad \begin{array}{c}
V \otimes V \\
\downarrow \\
V \otimes V
\end{array}$$

$\mathcal{H}_k$ has a commuting action with $U_q\mathfrak{gl}_n$ on the space

$$L((b^d)) \otimes (L(\square)) \otimes L((a^c)) \quad \text{with} \ c, d < n$$
Exploring tensor space structure

Move the right pole to the left:

\[ N \otimes V \otimes V \otimes V \otimes V \otimes V \otimes M = M \otimes N \otimes V \otimes V \otimes V \otimes V \otimes V \otimes V \]

New favorite generators:

\[ T_0 = \text{[diagram]} \]

Then

\[ M \otimes N = \text{[diagram]} \]

where \( \Lambda \) is the following set of partitions:

\[ \begin{array}{c}
  a \\
  b \\
  c \\
  d
\end{array} \]
Exploring tensor space structure

Move the right pole to the left:

\[ N \otimes V \otimes V \otimes V \otimes V \otimes V \otimes M = M \otimes M \otimes V \otimes V \otimes V \otimes V \otimes V \]

New favorite generators:

\[ T_0 = \quad T_i = \quad \text{and} \quad Y_j = \]

Then

\[ M \otimes N \otimes V \otimes V \otimes V \otimes V \otimes V = \bigoplus_{\lambda \in \Lambda} L(\lambda), \quad \text{(multiplicity one!)} \]
Exploring tensor space structure

Move the right pole to the left:

\[ N \otimes V \otimes V \otimes V \otimes V \otimes V \otimes V \otimes V \otimes V \otimes M = M \otimes N \otimes V \otimes V \otimes V \otimes V \otimes V \otimes V \otimes V \otimes V \]

New favorite generators:

\[ T_0 = \begin{array}{c} \vdots \\ \vdots \end{array}, \quad T_i = \begin{array}{c} \vdots \\ \vdots \end{array} \quad \text{and} \quad Y_j = \begin{array}{c} \vdots \\ \vdots \end{array} \]

Then

\[ M \otimes N = L((a^c)) \otimes L((b^d)) = \bigoplus_{\lambda \in \Lambda} L(\lambda), \quad \text{(multiplicity one!)} \]

where \( \Lambda \) is the following set of partitions:
Exploring tensor space structure

Move the right pole to the left:

\[ N \otimes V \otimes V \otimes V \otimes V \otimes V \otimes V \otimes M = M \otimes N \otimes V \otimes V \otimes V \otimes V \otimes V \otimes V \]

New favorite generators:

\[ T_0 = , \quad T_i = \quad \text{and} \quad Y_j = \]

Then

\[ M \otimes N = L((a^c)) \otimes L((b^d)) = \bigoplus_{\lambda \in \Lambda} L(\lambda), \quad \text{(multiplicity one!)} \]

where \( \Lambda \) is the following set of partitions:
Exploring tensor space structure

Move the right pole to the left:

\[
\begin{array}{ccc}
N \otimes V & \otimes & V \otimes V \otimes V \otimes V \otimes V \otimes V \otimes M \\
& & \Downarrow \\
N \otimes V & \otimes & V \otimes V \otimes V \otimes V \otimes V \otimes V \otimes M \\
\end{array}
= \begin{array}{ccc}
M \otimes N \otimes V & \otimes & V \otimes V \otimes V \otimes V \otimes V \\
& & \\
M \otimes N \otimes V & \otimes & V \otimes V \otimes V \otimes V \otimes V
\end{array}
\]

New favorite generators:

\[
T_0 = \quad T_i = \quad \text{and} \quad Y_j = \quad \text{and}
\]

Then

\[
M \otimes N = L((a^c)) \otimes L((b^d)) = \bigoplus_{\lambda \in \Lambda} L(\lambda), \quad \text{(multiplicity one!)}
\]

where \( \Lambda \) is the following set of partitions:
Exploring tensor space structure

Move the right pole to the left:

\[
N \otimes V \otimes V \otimes V \otimes V \otimes V \otimes V \otimes M = M \otimes N \otimes V \otimes V \otimes V \otimes V \otimes V \otimes V
\]

New favorite generators:

\[
T_0 = \begin{array}{c}
\end{array}, \quad T_i = \begin{array}{c}
\end{array} \quad \text{and} \quad Y_j = \begin{array}{c}
\end{array}
\]

Then

\[
M \otimes N = L((a^c)) \otimes L((b^d)) = \bigoplus_{\lambda \in \Lambda} L(\lambda), \quad \text{(multiplicity one!)}
\]

where \( \Lambda \) is the following set of partitions:
Exploring tensor space structure

Move the right pole to the left:

\[ N \otimes V \otimes V \otimes V \otimes V \otimes V \otimes M = M \otimes N \otimes V \otimes V \otimes V \otimes V \otimes V \]

New favorite generators:

\[ T_0 = \quad T_i = \quad \text{and} \quad Y_j = \]

Then

\[ M \otimes N = L((a^c)) \otimes L((b^d)) = \bigoplus_{\lambda \in \Lambda} L(\lambda), \quad \text{(multiplicity one!)} \]

where \( \Lambda \) is the following set of partitions...
Exploring tensor space structure
Exploring tensor space structure

\[ a \quad c \quad \ldots \quad M \quad k = 0 \]
Exploring tensor space structure

\[ \alpha \]

\[ c \]

\[ M \]

\[ k = 0 \]
Exploring tensor space structure
$L \left( \begin{array}{c c c c c c} 1 & 2 & 3 & 4 & 5 & 6 \\ 7 & 8 & 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 & 17 & 18 \\ 19 & 20 & 21 & 22 & 23 & 24 \\ 25 & 26 & 27 & 28 & 29 & 30 \\ 31 & 32 & 33 & 34 & 35 & 36 \\ \end{array} \right) \otimes L \left( \begin{array}{c c} 1 & 2 \\ \end{array} \right)$
$L \left( \begin{array}{cccc} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{array} \right) \otimes L \left( \begin{array}{cccc} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{array} \right) \otimes L \left( \begin{array}{cccc} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{array} \right)$
$L \left( \begin{array}{c c c} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{array} \right) \otimes L \left( \begin{array}{c c c} 10 & 11 & 12 \\ 13 & 14 & 15 \\ 16 & 17 & 18 \end{array} \right) \otimes L (\square) \otimes L (\square)$

H$_k$ representations in tensor space are labeled by certain partitions $\lambda$, with basis labeled by tableaux from some partition $\mu$ in $(a \, c) \otimes (b \, d)$ to $\lambda$. Rep are calibrated, i.e. $Y_j$'s act by constants controlled by content.
\( L \left( \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & 4 & 5 \\ \hline \end{array} \right) \otimes L \left( \begin{array}{|c|c|c|} \hline 1 & 2 \\ \hline \end{array} \right) \otimes L (\square) \otimes L (\square) \otimes L (\square) \)
\[ L \left( \begin{array}{cccc} 1 & 2 & \cdots & r_1 \\ 3 & 4 & \cdots & r_2 \\ \vdots & \vdots & \ddots & \vdots \\ r_1 & r_2 & \cdots & 2 \\ 1 & & & 1 \end{array} \right) \otimes L \left( \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \right) \otimes L \left( \begin{array}{c} \square \end{array} \right) \otimes L \left( \begin{array}{c} \square \end{array} \right) \otimes L \left( \begin{array}{c} \square \end{array} \right) \otimes L \left( \begin{array}{c} \square \end{array} \right) \otimes L \left( \begin{array}{c} \square \end{array} \right) \]
$L \left( \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \\ \end{array} \right) \otimes L \left( \begin{array}{cc} 1 \\ 2 \\ \end{array} \right) \otimes L \left( \square \right) \otimes L \left( \square \right) \otimes L \left( \square \right) \otimes L \left( \square \right) \otimes L \left( \square \right) \otimes L \left( \square \right) \otimes L \left( \square \right) \otimes L \left( \square \right)$

The $Y_j$'s act by constants controlled by content.

Hence, representations in tensor space are labeled by certain partitions $\lambda$, with basis labeled by tableaux from some partition $\mu$ in $(a \ c) \otimes (b \ d) \otimes \lambda$. Rep are calibrated, i.e. boxes that must appear in the partition at level 0.
$L \left( \begin{array}{cccc} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \\ q & r & s & t \end{array} \right) \otimes L \left( \begin{array}{c} u \\ v \end{array} \right) \otimes L \left( \begin{array}{c} w \\ x \end{array} \right) \otimes L \left( \begin{array}{c} y \\ z \end{array} \right) \otimes L \left( \begin{array}{c} a' \\ b' \end{array} \right) \otimes L \left( \begin{array}{c} c' \\ d' \end{array} \right) \otimes L \left( \begin{array}{c} e' \\ f' \end{array} \right) \otimes L \left( \begin{array}{c} g' \\ h' \end{array} \right)$
$L \left( \begin{array}{c|c|c|c}
1 & 2 & 3 & 4 \\
4 & & & \\
& 5 & & \\
& & 2 & \\
& & & 3
\end{array} \right) \otimes L \left( \begin{array}{c|c|c|c}
1 & 2 & 3 & 4 \\
4 & & & \\
& 5 & & \\
& & 2 & \\
& & & 3
\end{array} \right) \otimes L (\square) \otimes L (\square) \otimes L (\square) \otimes L (\square) \otimes L (\square) \otimes L (\square)$

$\mathcal{H}_k$ representations in tensor space are labeled by certain partitions $\lambda$, with basis labeled by tableaux from some partition $\mu$ in $(a^c) \otimes (b^d)$ to $\lambda$. 
$L \left( \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \end{array} \right) \otimes L \left( \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \vdots \end{array} \right) \otimes L (\square) \otimes L (\square) \otimes L (\square) \otimes L (\square) \otimes L (\square) \otimes L (\square)$

$Y_j \mapsto t_{-j}.$  

$H_k$ representations in tensor space are labeled by certain partitions $\lambda$, with basis labeled by tableaux from some partition $\mu$ in $(a^c) \otimes (b^d)$ to $\lambda$. Rep are calibrated, i.e. $Y_j$'s act by constants controlled by content.
$L \left( \begin{array}{c|c|c|c|c|c|c} 
& & & & & & \\
& & & & & & \\
& & & & & & \\
& & & & & & \\
& & & & & & \\
& & & & & & \\
\end{array} \right) \otimes L \left( \begin{array}{c|c|c|c|c|c|c} 
& & & & & & \\
& & & & & & \\
& & & & & & \\
& & & & & & \\
& & & & & & \\
& & & & & & \\
\end{array} \right) \otimes L \left( \begin{array}{|c|c|c|c} 
& & & \\
& & & \\
& & & \\
& & & \\
\end{array} \right) \otimes L \left( \begin{array}{|c|c|c|c} 
& & & \\
& & & \\
& & & \\
& & & \\
\end{array} \right) \otimes L \left( \begin{array}{|c|c|c|c} 
& & & \\
& & & \\
& & & \\
& & & \\
\end{array} \right) \otimes L \left( \begin{array}{|c|c|c|c} 
& & & \\
& & & \\
& & & \\
& & & \\
\end{array} \right) \otimes L \left( \begin{array}{|c|c|c|c} 
& & & \\
& & & \\
& & & \\
& & & \\
\end{array} \right)$

Shift by $\frac{1}{2}(a - c + b - d)$

$\mathcal{H}_k$ representations in tensor space are labeled by certain partitions $\lambda$, with basis labeled by tableaux from some partition $\mu$ in $(a^c) \otimes (b^d)$ to $\lambda$. Rep are calibrated, i.e. $Y_j$'s act by constants controlled by content.
$L \left( \begin{array}{c}
L
\end{array} \right) \otimes L \left( \begin{array}{c}
\end{array} \right) \otimes L \left( \begin{array}{c}
\end{array} \right) \otimes L \left( \begin{array}{c}
\end{array} \right) \otimes L \left( \begin{array}{c}
\end{array} \right) \otimes L \left( \begin{array}{c}
\end{array} \right) \otimes L \left( \begin{array}{c}
\end{array} \right)$

Shift by $\frac{1}{2}(a - c + b - d)$

$Y_1 \leftrightarrow t^{5.5}$
$Y_2 \leftrightarrow t^{3.5}$
$Y_3 \leftrightarrow t^{-4.5}$
$Y_4 \leftrightarrow t^{-5.5}$
$Y_5 \leftrightarrow t^{-2.5}$

$\mathcal{H}_k$ representations in tensor space are labeled by certain partitions $\lambda$, with basis labeled by tableaux from some partition $\mu$ in $(a^c) \otimes (b^d)$ to $\lambda$. Rep are calibrated, i.e. $Y_j$'s act by constants controlled by content.
\[ \mathcal{H}_k \text{ representations in tensor space are labeled by certain partitions } \lambda, \]
\[ \text{with basis labeled by tableaux from some partition } \mu \text{ in } (a^c) \otimes (b^d) \text{ to } \lambda. \]
\[ \text{Rep are calibrated, i.e. } Y_j \text{'s act by constants controlled by content.} \]

\[ Y_1 \mapsto t^{5.5} \]
\[ Y_2 \mapsto t^{3.5} \]
\[ Y_3 \mapsto t^{-4.5} \]
\[ Y_4 \mapsto t^{-5.5} \]
\[ Y_5 \mapsto t^{-2.5} \]
$L \left( \begin{array}{c|c|c|c} 1 & 2 & 3 & 4 \\ \hline 5 & 6 & 7 & 8 \\ \hline 9 & 10 & 11 & 12 \\ \hline 13 & 14 & 15 & 16 \end{array} \right) \otimes L \left( \begin{array}{c|c|c|c} \\ \hline \\ \hline \\ \hline \\ \hline \\ \hline \end{array} \right) \otimes L \left( \begin{array}{c|c|c|c} \\ \hline \\ \hline \\ \hline \\ \hline \\ \hline \end{array} \right) \otimes L \left( \begin{array}{c|c|c|c} \\ \hline \\ \hline \\ \hline \\ \hline \\ \hline \end{array} \right) \otimes L \left( \begin{array}{c|c|c|c} \\ \hline \\ \hline \\ \hline \\ \hline \\ \hline \end{array} \right) \otimes L \left( \begin{array}{c|c|c|c} \\ \hline \\ \hline \\ \hline \\ \hline \\ \hline \end{array} \right)$

Shift by $\frac{1}{2}(a - c + b - d)$

\[
\begin{array}{cccccccc}
Y_1 & \mapsto & t^{5.5} \\
Y_2 & \mapsto & t^{3.5} \\
Y_3 & \mapsto & t^{-4.5} \\
Y_4 & \mapsto & t^{-5.5} \\
Y_5 & \mapsto & t^{-2.5} \\
Y_6 & \mapsto & t^{5.5} \\
Y_7 & \mapsto & t^{2.5} \\
Y_8 & \mapsto & t^{4.5} \\
Y_9 & \mapsto & t^{3.5}
\end{array}
\]

$\mathcal{H}_k$ representations in tensor space are labeled by certain partitions $\lambda$, with basis labeled by tableaux from some partition $\mu$ in $(a^c) \otimes (b^d)$ to $\lambda$. Rep are calibrated, i.e. $Y_j$'s act by constants controlled by content.
\[ L \left( \begin{array}{c|c|c|c|c|c|c} \hline & & & & & & \\ \hline \hline & & & & & & \\ \hline \hline & & & & & & \\ \hline \hline & & & & & & \\ \hline \end{array} \right) \otimes L \left( \begin{array}{c|c} \hline & \\
\hline \end{array} \right) \otimes L \left( \begin{array}{c|c} \hline & \\
\hline \end{array} \right) \otimes L \left( \begin{array}{c|c} \hline & \\
\hline \end{array} \right) \otimes L \left( \begin{array}{c|c} \hline & \\
\hline \end{array} \right) \otimes L \left( \begin{array}{c|c} \hline & \\
\hline \end{array} \right) \otimes L \left( \begin{array}{c|c} \hline & \\
\hline \end{array} \right) \]

Shift by \( \frac{1}{2}(a - c + b - d) \)

\[
\begin{align*}
Y_1 & \mapsto t^{5.5} \\
Y_2 & \mapsto t^{3.5} \\
Y_3 & \mapsto t^{-4.5} \\
Y_4 & \mapsto t^{-5.5} \\
Y_5 & \mapsto t^{-2.5}
\end{align*}
\]

\[ \mathcal{H}_k \] representations in tensor space are labeled by certain partitions \( \lambda \), with basis labeled by tableaux from some partition \( \mu \) in \( (a^c) \otimes (b^d) \) to \( \lambda \). Rep are calibrated, i.e. \( Y_j \)'s act by constants controlled by content.

boxes that must appear in the partition at level 0.
\( L \left( \begin{array}{ccc} \lambda_1 & \lambda_2 & \lambda_3 \end{array} \right) \otimes L \left( \lambda_4 \right) \otimes L \left( \lambda_5 \right) \otimes L \left( \lambda_6 \right) \otimes L \left( \lambda_7 \right) \otimes L \left( \lambda_8 \right) \otimes L \left( \lambda_9 \right) \)

Shift by \( \frac{1}{2} (a - c + b - d) \)

\[ \begin{align*}
Y_1 & \mapsto t^{5.5} \\
Y_2 & \mapsto t^{3.5} \\
Y_3 & \mapsto t^{-4.5} \\
Y_4 & \mapsto t^{-5.5} \\
Y_5 & \mapsto t^{-2.5}
\end{align*} \]

\[ \begin{align*}
Y_1 & \mapsto t^{-5.5} \\
Y_2 & \mapsto t^{2.5} \\
Y_3 & \mapsto t^{4.5} \\
Y_4 & \mapsto t^{3.5} \\
Y_5 & \mapsto t^{5.5}
\end{align*} \]

\( \mathcal{H}_k \) representations in tensor space are labeled by certain partitions \( \lambda \), with basis labeled by tableaux from some partition \( \mu \) in \((a^c) \otimes (b^d)\) to \( \lambda \). Rep are calibrated, i.e. \( Y_j \)'s act by constants controlled by content.

boxes that must appear in the partition at level 0.
\[ L \left( \begin{array}{ccc} & & \\ & & \\ & & \end{array} \right) \otimes L \left( \begin{array}{cc} & \\ & \end{array} \right) \otimes L \left( \begin{array}{c} \\ \end{array} \right) \otimes L \left( \begin{array}{c} \\ \end{array} \right) \otimes L \left( \begin{array}{c} \\ \end{array} \right) \otimes L \left( \begin{array}{c} \\ \end{array} \right) \otimes L \left( \begin{array}{c} \\ \end{array} \right) \]

**Shift by** $\frac{1}{2}(a - c + b - d)$

\[
Y_1 \mapsto t^{5.5} \\
Y_2 \mapsto t^{3.5} \\
Y_3 \mapsto t^{-4.5} \\
Y_4 \mapsto t^{-5.5} \\
Y_5 \mapsto t^{-2.5}
\]

\[ H_k \] representations in tensor space are labeled by certain partitions $\lambda$, with basis labeled by tableaux from some partition $\mu$ in $(a^c) \otimes (b^d)$ to $\lambda$. Rep are calibrated, i.e. $Y_j$’s act by constants controlled by content.

boxes that must appear in the partition at level 0.
\[ L \left( \begin{array}{c} \hline \hline \end{array} \right) \otimes L \left( \begin{array}{c} \hline \hline \end{array} \right) \otimes L \left( \begin{array}{c} \hline \end{array} \right) \otimes L \left( \begin{array}{c} \hline \end{array} \right) \otimes L \left( \begin{array}{c} \hline \end{array} \right) \otimes L \left( \begin{array}{c} \hline \end{array} \right) \otimes L \left( \begin{array}{c} \hline \end{array} \right) \]

Shift by \( \frac{1}{2}(a - c + b - d) \)

\[
\begin{array}{cccccccc}
Y_1 & \mapsto & t^{5.5} \\
Y_2 & \mapsto & t^{3.5} \\
Y_3 & \mapsto & t^{-4.5} \\
Y_4 & \mapsto & t^{-5.5} \\
Y_5 & \mapsto & t^{-2.5} \\
\end{array}
\]

\( \mathcal{H}_k \) representations in tensor space are labeled by certain partitions \( \lambda \), with basis labeled by tableaux from some partition \( \mu \) in \( (a^c) \otimes (b^d) \) to \( \lambda \). Rep are calibrated, i.e. \( Y_j \)'s act by constants controlled by content.

boxes that must appear in the partition at level 0.
\[
L \left( \begin{array}{c|c|c|c|c|c|c}
\hline
& & & & & & \\
\hline
& & & & & & \\
\hline
& & & & & & \\
\hline
& & & & & & \\
\hline
& & & & & & \\
\end{array} \right) \otimes L \left( \begin{array}{c|c|c|c|c|c|c}
\hline
& & & & & & \\
\hline
& & & & & & \\
\hline
& & & & & & \\
\hline
& & & & & & \\
\hline
& & & & & & \\
\end{array} \right) \otimes L (\square) \otimes L (\square) \otimes L (\square) \otimes L (\square) \otimes L (\square) \otimes L (\square)
\]

Shift by \( \frac{1}{2} (a - c + b - d) \)

\[
Y_1 \mapsto t^{5.5} \\
Y_2 \mapsto t^{3.5} \\
Y_3 \mapsto t^{-4.5} \\
Y_4 \mapsto t^{-5.5} \\
Y_5 \mapsto t^{-2.5}
\]

\( \mathcal{H}_k \) representations in tensor space are labeled by certain partitions \( \lambda \), with basis labeled by tableaux from some partition \( \mu \) in \((a^c) \otimes (b^d)\) to \( \lambda \). Rep are calibrated, i.e. \( Y_j \)'s act by constants controlled by content.

boxes that must appear in the partition at level 0.
\( L \left( \begin{array}{cccc} & \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet & \bullet \\ \end{array} \right) \otimes L \left( \begin{array}{c} \bullet \\ \hline \bullet \\ \end{array} \right) \otimes L \left( \begin{array}{c} \bullet \\ \hline \bullet \\ \end{array} \right) \otimes L \left( \begin{array}{c} \bullet \\ \hline \bullet \\ \end{array} \right) \otimes L \left( \begin{array}{c} \bullet \\ \hline \bullet \\ \end{array} \right) \otimes L \left( \begin{array}{c} \bullet \\ \hline \bullet \\ \end{array} \right) \otimes L \left( \begin{array}{c} \bullet \\ \hline \bullet \\ \end{array} \right) \)

Shift by \( \frac{1}{2}(a - c + b - d) \)

\[
\begin{align*}
Y_1 & \mapsto t^{5.5} \\
Y_2 & \mapsto t^{3.5} \\
Y_3 & \mapsto t^{-4.5} \\
Y_4 & \mapsto t^{-5.5} \\
Y_5 & \mapsto t^{-2.5}
\end{align*}
\]

\( \mathcal{H}_k \) representations in tensor space are labeled by certain partitions \( \lambda \), with basis labeled by tableaux from some partition \( \mu \) in \((a^c) \otimes (b^d)\) to \( \lambda \). Rep are calibrated, i.e. \( Y_j \)'s act by constants controlled by content.

Boxes that must appear in the partition at level 0.
\( L \left( \begin{array}{|c|c|c|c|} \hline & & & \\
\hline & & & \\
\hline & & & \\
\hline \end{array} \right) \otimes L \left( \begin{array}{|c|c|} \hline & \\
\hline \end{array} \right) \otimes L \left( \begin{array}{|c|} \hline \\
\hline \end{array} \right) \otimes L \left( \begin{array}{|c|} \hline \\
\hline \end{array} \right) \otimes L \left( \begin{array}{|c|} \hline \\
\hline \end{array} \right) \otimes L \left( \begin{array}{|c|} \hline \\
\hline \end{array} \right) \otimes L \left( \begin{array}{|c|} \hline \\
\hline \end{array} \right) \)

Shift by \( \frac{1}{2}(a-c+b-d) \)

\[
\begin{align*}
Y_1 & \mapsto t^{5.5} \\
Y_2 & \mapsto t^{3.5} \\
Y_3 & \mapsto t^{-4.5} \\
Y_4 & \mapsto t^{-5.5} \\
Y_5 & \mapsto t^{-2.5}
\end{align*}
\]

\[ Y_1 \mapsto t^{-5.5} \\
Y_2 \mapsto t^{2.5} \\
Y_3 \mapsto t^{4.5} \\
Y_4 \mapsto t^{3.5} \\
Y_5 \mapsto t^{5.5} \]

\[ \mathcal{H}_k \text{ representations in tensor space are labeled by certain partitions } \lambda, \]
with basis labeled by tableaux from some partition \( \mu \) in \( (a^c) \otimes (b^d) \) to \( \lambda \).

Rep are calibrated, i.e. \( Y_j \)'s act by constants controlled by content.

boxes that must appear in the partition at level 0.
Central characters

The Hecke algebra $\mathcal{H}_k$ features invertible, pairwise commuting elements $Y_1, \ldots, Y_k$ (weight lattice part).

\[
Y_j = \begin{array}{c}
|\hspace{-0.2em}|
|\hspace{-0.2em}|
|\hspace{-0.2em}|
|\hspace{-0.2em}|
|\hspace{-0.2em}|
|\hspace{-0.2em}|
\end{array}
\]
Central characters

The Hecke algebra $\mathcal{H}_k$ features invertible, pairwise commuting elements $Y_1, \ldots, Y_k$ (weight lattice part). The Weyl group $\mathcal{W}$ of type C (the group of signed permutations) acts on $\mathbb{C}[Y_1^{\pm 1}, \ldots, Y_k^{\pm 1}]$ by permuting the subscripts, with $Y_{-i} = Y_i^{-1}$. Then the center of $\mathcal{H}_k$ is symmetric Laurent polynomials

$$Z(\mathcal{H}_k) = \mathbb{C}[Y_1^{\pm 1}, \ldots, Y_k^{\pm 1}]^\mathcal{W}.$$ 

$$Y_j = \begin{array}{c} \bullet \end{array} \begin{array}{c} j \end{array}$$
Central characters

The Hecke algebra $\mathcal{H}_k$ features invertible, pairwise commuting elements $Y_1, \ldots, Y_k$ (weight lattice part).

The Weyl group $\mathcal{W}$ of type C (the group of signed permutations) acts on $\mathbb{C}[Y_1^{\pm 1}, \ldots, Y_k^{\pm 1}]$ by permuting the subscripts, with $Y_{-i} = Y_i^{-1}$. Then the center of $\mathcal{H}_k$ is symmetric Laurent polynomials

$$Z(\mathcal{H}_k) = \mathbb{C}[Y_1^{\pm 1}, \ldots, Y_k^{\pm 1}]^{\mathcal{W}}.$$

We can encode central characters as maps

$$\gamma : \{Y_1^{\pm 1}, \ldots, Y_k^{\pm 1}\} \rightarrow \mathbb{C}^\times$$

with equivalence under $\mathcal{W}$ action;
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$$\gamma = (\gamma_1, \ldots, \gamma_k) \quad \text{with} \quad \gamma(Y_i^{\pm 1}) = (\gamma_i)^{\pm 1}$$
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$$\mathbf{c} = (c_1, \ldots, c_k) \quad \text{with} \quad \gamma(Y_i^{\pm 1}) = t^{\pm c_i}$$

(when $\mathbf{c}$ is real, favorite representatives satisfy $0 \leq c_1 \leq \cdots \leq c_k$.)
Central characters as points

Fav equivalence class reps: \(0 \leq c_1 \leq \cdots \leq c_k\).

When \(k = 2\):

\[
\begin{align*}
(c_1, c_2) &= (c_1, c_1) \quad (c_1, c_2) \\
&= \begin{cases} 
  c_1, & \text{for } c_1 = c_2 \\
  c_2, & \text{for } c_1 \neq c_2
\end{cases}
\end{align*}
\]

\(c_1 = c_2\)
Central characters as points

Fav equivalence class reps: $0 \leq c_1 \leq \cdots \leq c_k$.

When $k = 2$:

$$c_1 = c_2 \quad (c_1, c_2)$$

The $r_i$'s depend on $H_k$'s parameters $t_0$ and $t_k$:

$$r_1 = \log \left( \frac{t_0}{t_k} \right), \quad r_2 = \log \left( \frac{t_0 t_k}{t} \right)$$
Central characters as points

Fav equivalence class reps: $0 \leq c_1 \leq \cdots \leq c_k$.

When $k = 2$:

$\mathfrak{h}^{\alpha_1 + \alpha_2}$  \hspace{2cm} $\mathfrak{h}^{\alpha_2}$  \hspace{2cm} $\mathfrak{h}^{\alpha_1}$

$c_2 = c_1 + 1$

Restrict to real points.
Central characters as points

Fav equivalence class reps: \( 0 \leq c_1 \leq \cdots \leq c_k \).

When \( k = 2 \):

\[
\begin{align*}
\mathfrak{h}^{\alpha_1 + \alpha_2} & \quad \mathfrak{h}^{\alpha_2} & \quad \mathfrak{h}^{\alpha_1} \\
\mathfrak{h}^{\alpha_2 + 2\alpha_1} & \quad \mathfrak{h}^{\alpha_2} & \quad \mathfrak{h}^{\alpha_2 + 2\alpha_1}
\end{align*}
\]

\( c_2 = c_1 + 1 \)
\( c_2 = c_1 - 1 \)
\( c_2 = -c_1 + 1 \)

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\(c_2 = r_2\)
\(c_2 = r_1\)
\(c_2 = c_1 + 1\)
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\(c_1 = r_1\)
\(c_1 = r_2\)

\(c_2 = -c_1 + 1\)
\(c_2 = -c_1 - 1\)

The \(r_i\)'s depend on \(\mathcal{H}_k\)'s parameters \(t_0\) and \(t_k\): \(r_1 = \log_t(t_0/t_k), \quad r_2 = \log_t(t_0t_k)\)

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Central characters as points; 
Calibrated reps as “skew local regions” 

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\[ c_2 = r_2 \quad c_2 = r_1 \quad c_2 = c_1 + 1 \quad c_2 = -c_1 + 1 \]

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Central characters as points; 
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\[ c_1 = r_1 \]  
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Thm. (D.-Ram)

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\[ k + a + b - \ell \]