Centralizers of the infinite symmetric group

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Schur-Weyl duality – a warm-up

Start with the symmetric group $S_k$: permutations of $1, \ldots, k$. Depict using permutation diagrams:
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1. $\text{GL}_n(\mathbb{C})$ acts on $\mathbb{C}^n \otimes \mathbb{C}^n \otimes \cdots \otimes \mathbb{C}^n = (\mathbb{C}^n)^\otimes k$ diagonally.

\[ g \cdot (v_1 \otimes v_2 \otimes \cdots \otimes v_k) = gv_1 \otimes gv_2 \otimes \cdots \otimes gv_k. \]
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2. $S_k$ also acts on $(\mathbb{C}^n)^{\otimes k}$ by place permutation.

3. These actions commute!

$$gv_3 \otimes gv_1 \otimes gv_5 \otimes gv_2 \otimes gv_4$$

vs.

$$gv_1 \otimes gv_2 \otimes gv_3 \otimes gv_4 \otimes gv_5$$
Schur-Weyl duality – a warm-up

**Schur-Weyl duality**: $S_k$ and $GL_n$ have commuting actions on $(C^n)^\otimes k$, and their images fully centralize each in $End\left( (C^n)^\otimes k \right)$. Why this is exciting: Huge transfer of information! Centralizer relationship produces $(C^n)^\otimes k \cong \bigoplus \lambda \vdash k \ G_\lambda \otimes S_\lambda$ as a $GL_n$-$S_k$ bimodule, where $G_\lambda$ are distinct irreducible $GL_n$-modules and $S_\lambda$ are distinct irreducible $S_k$-modules.

For example, $C^n \otimes C^n \otimes C^n \cong (G \otimes S) \oplus (G \otimes S) \oplus \ldots$
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For example,

$$\mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n \cong \left( G^\square \otimes S^\square \right) \oplus \left( G^\diamondsuit \otimes S^\diamondsuit \right) \oplus \left( G^\heartsuit \otimes S^\heartsuit \right)$$
Switching roles: the partition algebra

Let $V$ be the permutation representation of $S_n$:

$n \times n$ matrices with 1's and 0's \hspace{1cm} \text{i.e.} \hspace{1cm} \sigma \cdot v_i = v_{\sigma(i)}$

Now let $S_n$ act diagonally on $V^\otimes k$:

$\sigma \cdot (v_{i_1} \otimes \cdots \otimes v_{i_k}) = v_{\sigma(i_1)} \otimes \cdots \otimes v_{\sigma(i_k)}$
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$$\delta_{a=b=c} \quad (v_a \otimes v_a) \otimes \left( \sum_{i=1}^{n} v_i \otimes v_i \right)$$

\[ \uparrow \varphi \]
Set partitions

Fix $k \in \mathbb{Z}_{>0}$, and let

$$[k] = \{1, \ldots, k\} \quad \text{and} \quad [k'] = \{1', \ldots, k'\}.$$
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We’re interested in set partitions of \([k] \cup [k']\).

\[
\begin{array}{c}
\{1,2\} \\
\{3\} \\
\{2',3',4',4\}
\end{array}
\]
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(Both encode the map $v_a \otimes v_b \otimes v_c \otimes v_d \mapsto \delta_{b=c=d}(v_a \otimes v_a) \otimes \sum_{i=1}^{n} v_i \otimes v_b)$
The partition algebra

Multiplying diagrams:

\[ \begin{array}{ccc}
1 & 2 \\
\bullet & \bullet \\
1' & 2' \\
\end{array} \quad \begin{array}{c}
3 \\
\bullet \\
3' \\
\end{array} \quad \begin{array}{c}
4 \\
\bullet \\
4' \\
\end{array} \]

\[ \begin{array}{c}
3 \\
\bullet \\
3' \\
\end{array} \quad \begin{array}{c}
4 \\
\bullet \\
4' \\
\end{array} \]

The partition algebra \( P_k(n) \) is the \( C \)-span of the partition diagrams with this product.

Nice facts:

- \( \ast \) Associative algebra with identity \( 1 = \{1,1'\}, \ldots, \{k,k'\} \).
- \( \ast \) \( \dim(P_k(n)) = \) the Bell number \( B(2^k) \).
- \( \ast \) \( S_n \) and \( P_k(n) \) centralize each other in \( \text{End}(V \otimes k) \).
The partition algebra

Multiplying diagrams:

\[
1 \quad 2 \quad 3 \quad 4 \\
1' \quad 2' \quad 3' \quad 4' \\
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\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
\bullet & \bullet & \bullet & \bullet \\
\vdots & \vdots & \vdots & \vdots \\
1'' & 2'' & 3'' & 4'' \\
\end{array}
\]

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Multiplying diagrams:

\[
\begin{align*}
1 & \quad 2 & \quad 3 & \quad 4 \\
1'' & \quad 2'' & \quad 3'' & \quad 4'' \\
\vdots & \quad \vdots & \quad \vdots & \quad \vdots \\
\end{align*}
\]

\[
\begin{align*}
1 & \quad 2 & \quad 3 & \quad 4 \\
1' & \quad 2' & \quad 3' & \quad 4' \\
\end{align*}
\]

\[
= n^1
\]
The partition algebra

Multiplying diagrams:

\[ \begin{array}{cccc}
1 & 2 & 3 & 4 \\
\cdot & \cdot & \cdot & \cdot \\
1^{\prime} & 2^{\prime} & 3^{\prime} & 4^{\prime} \\
\cdot & \cdot & \cdot & \cdot \\
1^{\prime\prime} & 2^{\prime\prime} & 3^{\prime\prime} & 4^{\prime\prime}
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A connection to symmetric functions

As a consequence of the commuting relationship, the $S_n$-invariants in $V^\otimes k$ form a natural $P_k(n)$-module.
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\[ \bullet \bullet \bullet \leftrightarrow m\{\{1\},\{2\},\{3\}\} = \sum_{1\leq a,b,c \leq n} v_a \otimes v_b \otimes v_c \]

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\bullet \bullet \\
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As a consequence of the commuting relationship, the \( S_n \)-invariants in \( V \otimes^k \) form a natural \( P_k(n) \)-module. In fact, a basis for the \( S_n \)-invariants is indexed by set partitions of \([k]\), i.e. half diagrams:

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\end{align*}
\]

And the action is still by concatenation:

\[
d:
\begin{align*}
\begin{array}{ccccccc}
\bullet & \quad \bullet & \quad \bullet & \quad \bullet & \quad \bullet & \quad \bullet & \quad \bullet \\
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\[ \text{d:} \quad m\{\{1\},\{2\},\{3,4\}\} \quad \rightarrow \quad n^2 \quad m\{\{1,2\},\{3\},\{4\}\} \]
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Identify $V^\otimes k$ with the homogeneous degree-$k$ elements of $\mathbb{C}[x_1, \ldots, x_n]$ with non-commuting variables, via $v_a \leftrightarrow x_a$: 

So $P_k(n)$ acts on the degree-$k$ homogeneous elements of $\mathbb{C}[x_1, \ldots, x_n]$. 

Hopf algebras and symmetric functions:


($\ast$) Malvenuto-Reutenauer (1995) use Schur-Weyl duality to connect the graded Hopf algebra $\bigoplus k \mathbb{C}S_k$ to other classes of symmetric functions.

($\ast$) Aguiar-Orellana (2008) generalize [MR95] to connect a bigger diagram Hopf algebra (of uniform block permutations) and symmetric functions in non-commuting variables.

Issue: finitely many versus countably many variables!
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Issue: finitely many versus countably many variables!
Moving to the infinite symmetric group

Natural inclusion: \( S_n \subset S_{n+1} \) as permutations fixing \( n + 1 \).
Consider the limit

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S_1 \hookrightarrow S_2 \hookrightarrow S_3 \hookrightarrow \cdots \rightarrow S_\infty,
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so that \( S_\infty \) is the group of bijections on \( \mathbb{N} \) which fix all but finitely many elements.
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Want:

(1) A vector space \( V \) containing a countable linearly independent subset \( \{v_i\}_{i \in \mathbb{N}} \);
(2) an appropriate notion of \( V \otimes^k \); and
(3) an algebra of endomorphisms on \( V \otimes^k \)
   i. whose elements are determined by their image on \( v_i \)'s, and
   ii. which contains \( S_\infty \) via the above action.
Three examples explored:

1. Countable dimensional vector space \( V = \mathbb{C}^{(\mathbb{N})} = \mathbb{C}\{v_1, v_2, \ldots\} \).

2. Banach space of \( p \)-power summable sequences

   \[ V = \{ v = (a_1, a_2, \ldots) \in \mathbb{C}^\mathbb{N} \mid ||v||_p < \infty \} \]

3. Banach space of \( \ell^\infty \) bounded sequences

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   **Good:** Well-behaved vector space.
   
   **Bad:** No non-trivial $S_\infty$ invariants! e.g. if $k = 1, \sum_i v_i \notin V$)
   (Sam-Snowden 2013: representation theoretic stability)

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   - Bad: Must restrict to bounded maps, yielding restrictive results.

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3. Banach space of \( \ell^\infty \) bounded sequences

   \[ V = \{ v = (a_1, a_2, \ldots) \in \mathbb{C}^\mathbb{N} \mid \|v\|_\infty < \infty \} \] .

   \textbf{Good:} Has all the \( S_{\infty} \) invariants, and admits a larger set of maps.
   
   \textbf{Bad:} Even bounded maps aren’t well-behaved for our purposes.
Three examples explored:

1. Countable dimensional vector space $V = \mathbb{C}^{(\mathbb{N})} = \mathbb{C}\{v_1, v_2, \ldots\}$.
   
   **Good:** Well-behaved vector space.
   
   **Bad:** No non-trivial $S_\infty$ invariants! e.g. if $k = 1, \sum_i v_i \notin V$)
   
   (Sam-Snowden 2013: representation theoretic stability)

2. Banach space of $p$-power summable sequences

   $$V = \{v = (a_1, a_2, \ldots) \in \mathbb{C}^\mathbb{N} \mid \|v\|_p < \infty\}.$$  

   **Good:** Can get all the necessary $S_\infty$ invariants in each degree.
   
   **Bad:** Must restrict to bounded maps, yielding restrictive results.

3. Banach space of $\ell^\infty$ bounded sequences

   $$V = \{v = (a_1, a_2, \ldots) \in \mathbb{C}^\mathbb{N} \mid \|v\|_\infty < \infty\}.$$  

   **Good:** Has all the $S_\infty$ invariants, and admits a larger set of maps.
   
   **Bad:** Even bounded maps aren’t well-behaved for our purposes.

**Good/Bad?**

2 and 3 yield non-unitary and non-semisimple representations!
1. Countable dimensional vector space $V = \mathbb{C}^{(\mathbb{N})}$

If the $\varphi \in \text{End}(V \otimes k)$ commutes with the action of $S_\infty$, it acts like a linear combination of partition algebra diagrams.

Additionally, to be in $\text{End}(V \otimes k)$, its image must be a finite linear combination of $v_i$'s.
1. Countable dimensional vector space $V = \mathbb{C}^{(\mathbb{N})}$

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If the $\varphi \in \text{End}(V^\otimes k)$ commutes with the action of $S_\infty$, it acts like a linear combination of partition algebra diagrams.

Additionally, to be in $\text{End}(V^\otimes k)$, its image must be a finite linear combination of $v_i$'s.

\begin{align*}
\varphi &\uparrow & v_a \otimes v_a \\
v_a \otimes v_b & & \delta_{a,b} \left( v_a \otimes \left( \sum_i v_i \right) \right) \\
\varphi &\uparrow & v_a \otimes v_b
\end{align*}
1. Countable dimensional vector space \( V = \mathbb{C}^{(\mathbb{N})} \)

If the \( \varphi \in \text{End}(V^{\otimes k}) \) commutes with the action of \( S_\infty \), it acts like a linear combination of partition algebra diagrams.

Additionally, to be in \( \text{End}(V^{\otimes k}) \), its image must be a finite linear combination of \( v_i \)'s.

Yes! \[ \varphi \uparrow \]

No! \[ \delta_{a,b} \left( v_a \otimes \left( \sum_i v_i \right) \right) \]
1. Countable dimensional vector space \( V = \mathbb{C}^{(\mathbb{N})} \)

If the \( \varphi \in \text{End}(V \otimes k) \) commutes with the action of \( S_\infty \), it acts like a linear combination of partition algebra diagrams.

Additionally, to be in \( \text{End}(V \otimes k) \), its image must be a finite linear combination of \( v_i \)'s.

\[
\begin{align*}
\varphi (v_a \otimes v_a) & \quad \text{Yes!} \\
\varphi \left( v_a \otimes \left( \sum_i v_i \right) \right) & \quad \text{No!}
\end{align*}
\]

Result: The *top-propagating partition algebra*, generated by diagrams with no block isolated to the top.
(Sam-Snowden: the *upward partition category* glues all \( k \) together)
2. Banach space of $p$-power summable sequences

Place a metric $\mu$ on $\mathbb{C}^N$ so that
\[
\left\| \sum_i v_i \right\|_p = \left\| (1, 1, 1, \ldots) \right\|_p = \sum_i \mu_i^p < \infty.
\]
(Enough to get all expected invariants in the closure of $V^\otimes k$ for each $k$.)
2. Banach space of $p$-power summable sequences

Place a metric $\mu$ on $\mathbb{C}^\mathbb{N}$ so that
\[ \| \sum_i v_i \|^p_p = \| (1, 1, 1, \ldots) \|^p_p = \sum_i \mu_i^p < \infty. \]
(Enough to get all expected invariants in the closure of $V \otimes^k$ for each $k$.)

We restrict to continuous/bounded endomorphisms $\mathcal{B}(V \otimes^k)$. 

Yes:

No:

(Same algebra as in Aguiar-Orellana!)
2. Banach space of \( p \)-power summable sequences

Place a metric \( \mu \) on \( \mathbb{C}^\mathbb{N} \) so that
\[
\left\| \sum_i v_i \right\|_p = \left\| (1, 1, 1, \ldots) \right\|_p = \sum_i \mu_i^p < \infty.
\]
(Enough to get all expected invariants in the closure of \( V \otimes^k \) for each \( k \).)

We restrict to continuous/bounded endomorphisms \( B(V \otimes^k) \).

Again, if the \( \varphi \in B(V \otimes^k) \) commutes with \( S_\infty \), it acts like a linear combination of partition algebra diagrams.
Place a metric $\mu$ on $\mathbb{C}^N$ so that

$$\| \sum_i v_i \|_p = \|(1, 1, 1, \ldots)\|_p = \sum_i \mu_i^p < \infty.$$ (Enough to get all expected invariants in the closure of $V \otimes^k$ for each $k$.)

We restrict to continuous/bounded endomorphisms $\mathcal{B}(V \otimes^k)$.

Again, if the $\varphi \in \mathcal{B}(V \otimes^k)$ commutes with $S_\infty$, it acts like a linear combination of partition algebra diagrams.

However, boundedness then additionally restricts to maps whose image on simple tensors is a permutation of factors.

(All other partition diagrams have unbounded images).
2. Banach space of $p$-power summable sequences

Place a metric $\mu$ on $\mathbb{C}^N$ so that

$$\left\| \sum_i v_i \right\|_p = \left\| (1, 1, 1, \ldots) \right\|_p = \sum_i \mu_i^p < \infty.$$  

(Enough to get all expected invariants in the closure of $V^\otimes k$ for each $k$.)  

We restrict to continuous/bounded endomorphisms $B(\overline{V^\otimes k})$.

Again, if the $\varphi \in B(\overline{V^\otimes k})$ commutes with $S_\infty$, it acts like a linear combination of partition algebra diagrams.  

However, boundedness then additionally restricts to maps whose image on simple tensors is a permutation of factors.  

(All other partition diagrams have unbounded images).

**Result:** The algebra of *uniform block permutations*, generated by diagrams whose blocks have the same size on top as on bottom.

Yes: ![Yes Diagram](image1)  

No: ![No Diagram](image2)  

(Same algebra as in Aguiar-Orellana!)
3. Banach space of $\ell^\infty$-bounded sequences

Sequences $(a_1, a_2, \ldots) \in \mathbb{C}^\mathbb{N}$ whose entries are bounded.

Issue: Even $\ell^\infty$-bounded endomorphisms are not determined by their images on $\{v_i\}_{i \in \mathbb{N}}$.
3. Banach space of $\ell^\infty$-bounded sequences

Sequences $(a_1, a_2, \ldots) \in \mathbb{C}^\mathbb{N}$ whose entries are bounded.

**Issue:** Even $\ell^\infty$-bounded endomorphisms are not determined by their images on $\{v_i\}_{i \in \mathbb{N}}$. So let $B_{\text{Mat}}(\overline{V \otimes k})$ be the algebra of $\ell^\infty$-bounded maps which can be written as matrices.

(The sums across rows are $\ell_\infty$ bounded.)
3. Banach space of $\ell^\infty$-bounded sequences

Sequences $\left(a_1, a_2, \ldots \right) \in \mathbb{C}^\mathbb{N}$ whose entries are bounded.

**Issue:** Even $\ell^\infty$-bounded endomorphisms are not determined by their images on $\{v_i\}_{i \in \mathbb{N}}$. So let $\mathcal{B}_{\text{Mat}}(\overline{V \otimes k})$ be the algebra of $\ell^\infty$-bounded maps which can be written as matrices.

(The sums across rows are $\ell_\infty$ bounded.)

**Result:** The *bottom-propagating partition algebra*, generated by diagrams with no block isolated to the bottom. (Isomorphic to case 1)

![Yes:](attachment:yes_diagram.png)  ![No:](attachment:no_diagram.png)
Remark 1: Orellana et al. (in progress) show that if a diagram Hopf algebra (as in [MR95] or [AO08]) is built from partition diagrams, those diagrams can have no blocks isolated to the top or bottom rows.

Case 1: no application to symmetric functions.
Case 2: tied to symmetric functions in [AO08].
Question: Is there a fix for case 3?
Putting it back into context

**Remark 1:** Orellana et al. (in progress) show that if a diagram Hopf algebra (as in [MR95] or [AO08]) is built from partition diagrams, those diagrams can have no blocks isolated to the top or bottom rows.

Case 1: no application to symmetric functions.
Case 2: tied to symmetric functions in [AO08].
Question: Is there a fix for case 3?

**Remark 2:** For all three cases, even for $k = 1$, the centralizer algebra is spanned by $\mathbb{1}$, so is isomorphic to $\mathbb{C}$. However, in cases 2 and 3, we expected more since $V$ has an invariant subspace. This discrepancy comes from the fact that the action of $S_\infty$ is not semisimple.

Question: Can we use this framework to study certain non-unitary representations of $S_\infty$?