Combinatorics of affine Hecke algebras of type C.

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The affine type C Hecke algebra $H_k$ is generated by invertible elements $T_0, T_1, \ldots, T_k$ with relations

\[
T_0 \quad T_1 \quad T_2 \quad T_{k-2} \quad T_{k-1} \quad T_k
\]

\[
\bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet
\]

\[
(T_0 - t_0)(T_0 - t_0^{-1}) = 0 = (T_k - t_k)(T_k - t_k^{-1})
\]

\[
(T_i - t^{1/2})(T_i + t^{-1/2}) = 0 \text{ for } 1 \leq i \leq k - 1.
\]
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(T_0 - t_0)(T_0 - t_0^{-1}) &= 0 = (T_k - t_k)(T_k - t_k^{-1}) \\
(T_i - t^{1/2})(T_i + t^{-1/2}) &= 0 \text{ for } 1 \leq i \leq k - 1.
\end{align*}
$$

Instead, our favorite generators are $T_0, T_1, \ldots, T_{k-1}$ and invertible, pairwise commuting $Y_1, \ldots, Y_k$ (weight lattice part) with additional relations...
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\[(T_0 - t_0)(T_0 - t_0^{-1}) = 0 = (T_k - t_k)(T_k - t_k^{-1})\]
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Instead, our favorite generators are $T_0, T_1, \ldots, T_{k-1}$ and invertible, pairwise commuting $Y_1, \ldots, Y_k$ (weight lattice part) with additional relations…

**Goal today:**
Tell you 3 descriptions of calibrated irreducible reps of $H_k$, where “calibrated” means $\mathbb{C}[Y_1^{\pm 1}, \ldots, Y_k^{\pm 1}]$ is simultaneously diagonalized.
Central characters

The center of $H_k$ is symmetric Laurent polynomials

$$Z(H_k) = \mathbb{C}[Y_1^{\pm 1}, \ldots, Y_k^{\pm 1}]^{W_0}$$

w.r.t. the Weyl group $W_0$ of type C.
Central characters

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$$Z(H_k) = \mathbb{C}[Y_1^{\pm 1}, \ldots, Y_k^{\pm 1}]^{W_0}$$

w.r.t. the Weyl group $W_0$ of type C. We can encode central characters as maps

$$\gamma : \{Y_1^{\pm 1}, \ldots, Y_k^{\pm 1}\} \to \mathbb{C}$$

with equivalence under $W_0$ action;
Central characters

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$$\gamma = (\gamma_1, \ldots, \gamma_k) \quad \text{with} \quad \gamma(Y_i^\pm 1) = (\gamma_i)^\pm 1$$
Central characters

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$$\gamma = (\gamma_1, \ldots, \gamma_k) \quad \text{with} \quad \gamma(Y_i^{\pm 1}) = (\gamma_i)^{\pm 1}$$

$$c = (c_1, \ldots, c_k) \quad \text{with} \quad \gamma(Y_i^{\pm 1}) = t^{\pm c_i}$$

(where $W_0$ acts by signed permutations of $c$)
Central characters

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(where $W_0$ acts by signed permutations of $\mathbf{c}$)

**Description 1:** Central characters are indexed by points $\mathbf{c}$ in $\mathbb{C}^k$. 
Central characters as points

Fav equivalence class reps: $0 \leq c_1 \leq \cdots \leq c_k$. ($W_0$ acts by signed permutations)

When $k = 2$:
Central characters as points

Fav equivalence class reps: \( 0 \leq c_1 \leq \cdots \leq c_k \). \((W_0 \text{ acts by signed permutations})\)

When \( k = 2 \):

\[
\begin{align*}
&\mathfrak{h}^{\alpha_1 + \alpha_2} \\
&\mathfrak{h}^{\alpha_2} \\
&\mathfrak{h}^{\alpha_1} \\
&\mathfrak{h}^{\alpha_2 + 2\alpha_1}
\end{align*}
\]

\((c_1, c_2)\)
Central characters as points

Fav equivalence class reps: $0 \leq c_1 \leq \cdots \leq c_k$. ($W_0$ acts by signed permutations)

When $k = 2$:

$c_2 = c_1 + 1$
Central characters as points

Fav equivalence class reps: $0 \leq c_1 \leq \cdots \leq c_k$. ($W_0$ acts by signed permutations)

When $k = 2$:

- $c_2 = c_1 + 1$
- $c_2 = c_1 - 1$
- $c_2 = -c_1 + 1$
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Fav equivalence class reps: \(0 \leq c_1 \leq \cdots \leq c_k\). \((W_0 \text{ acts by signed permutations})\)

When \(k = 2\):

\[
\begin{align*}
\mathfrak{h}^{\alpha_1 + \alpha_2} & \quad \mathfrak{h}^{\alpha_2} \\
& \quad \mathfrak{h}^{\alpha_1}
\end{align*}
\]

\[
\begin{align*}
c_2 = r_2 & \quad \cdots \quad c_2 = -c_1 + 1 \\
c_2 = r_1 & \quad \cdots \quad c_2 = -c_1 - 1 \\
c_2 = c_1 + 1 \\
c_2 = c_1 - 1
\end{align*}
\]

The \(r_i\)'s depend on \(H_k\)'s parameters \(t_0\) and \(t_k\): \(r_1 = \log_t(t_0/t_k)\), \(r_2 = \log_t(t_0 t_k)\)
Central characters as points

Fav equivalence class reps: $0 \leq c_1 \leq \cdots \leq c_k$. ($W_0$ acts by signed permutations)

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Restrict to real points.
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When $k = 2$:

\[ h^{\alpha_1 + \alpha_2} \quad h^{\alpha_2} \quad h^{\alpha_1} \]

\[ h^{\alpha_2 + 2\alpha_1} \]

$c_2 = r_2$
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The $r_i$s depend on $H_k$'s parameters $t_0$ and $t_k$: $r_1 = \log_t(t_0 / t_k)$, $r_2 = \log_t(t_0 t_k)$
Central characters as points; Calibrated reps as skew local regions

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Description 1: Central characters are indexed by points \( c \) in \( \mathbb{C}^k \). Representations of \( H_k \) are indexed by skew local regions. Basis indexed by chambers.
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Description 2: Box arrangements.
**Description 1:** Central characters are indexed by points $c$ in $\mathbb{C}^k$. Representations of $H_k$ are indexed by skew local regions. Basis indexed by chambers.

**Description 2:** Box arrangements.
Restrict to $c_i \in \mathbb{Z} + \beta$ for some $\beta \in \mathbb{C}$.
A central character $c$ gives a list of diagonal placements.
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A central character \( c \) gives a list of diagonal placements. For example:

\[ c = (2, 3, 4, 4, 5) \]
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\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
\end{array}
\]

\[
\begin{array}{cccccc}
& & & & & \\
& & & & & \\
& & \textbf{3} & & & \\
& \textbf{2} & \textbf{4} & \textbf{5} & & \\
\textbf{1} & & & & & \\
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1  2  3  4  5  6
\[\begin{array}{cccccc}
\  &  &  &  &  & + \\
3 & 2 & 4 & 5 & \  & + \\
1 & & & & & \\
\end{array}\]

1 > 2, 2 > 3, 2 < 4, 3, 4 < 5

1  2  3  4  5  6
\[\begin{array}{cccccc}
\  &  &  &  &  & + \\
3 & 5 & & & & + \\
1 & 2 & 4 & & & \\
\end{array}\]

1 < 2, 2 > 3, 3 > 5, 4 > 5
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\[1 > 2, \ 2 > 3, \ 2 < 4, \ 3, 4 < 5\]

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Basis indexed by standard fillings with \( \{\pm 1, \ldots, \pm k\} \) with restrictions:
(1) Exactly one of \( i \) or \(-i\) appears.
(2) If \( \text{box}_i < \bullet \), then filling is negative. If \( \text{box}_i > \bullet \), filling is positive.
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1. Exactly one of $i$ or $-i$ appears.
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Points versus box arrangements

\[ h^\alpha_2 \quad c_1 = r_1 \quad c_1 = r_2 \quad h^\alpha_1 \]

\[ c_2 = r_2 \]

\[ c_2 = r_1 \]

\[ c_2 = c_1 + 1 \]
Points versus box arrangements

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\[ c_1 = r_2 \]

\[ c_1 = r_1 \]

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\[ c_2 = r_2 \]

\[ \begin{array}{|c|c|}
   \hline
   2 & 2 \\
   \hline
   1 & -1 \\
   \hline
   1 & -1 \\
   \hline
   2 & 2 \\
   \hline
   -2 & -2 \\
   \hline
   -1 & 1 \\
   \hline
   -2 & -2 \\
   \hline
   \end{array} \]
Points versus box arrangements

\[ c_2 = r_1 \quad c_1 = r_2 \]

\[ c_2 = c_1 + 1 \]
Points versus box arrangements

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Points versus box arrangements

\[ \mathcal{H}^{\alpha_2} \quad c_1 = r_1 \quad \mathcal{H}^{\alpha_1} \]

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Description 1: Central characters are indexed by points $c$ in $\mathbb{C}^k$. Representations of $H_k$ are indexed by skew local regions. Basis indexed by chambers.

Description 2: Marked box arrangements. Basis indexed by good fillings.
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**Description 2:** Marked box arrangements. Basis indexed by good fillings.

**Description 3:** Partitions. Representation arise in Schur-Weyl duality with certain $U_q gl_n$ reps.
Centralizer properties

Let $U = U_q \mathfrak{gl}_n$ be the quantum group for $\mathfrak{gl}_n(\mathbb{C})$. We’re interested in certain finite dimensional simple $U$-modules $L(\lambda)$ indexed by partitions:

$$\lambda = \begin{array}{cccc}
\hline
& & & \\
& & \\
& \\
\hline
\end{array}$$

(drawn as a collection of boxes piled up and to the left)
Centralizer properties

Let $U = U_q \mathfrak{gl}_n$ be the quantum group for $\mathfrak{gl}_n(\mathbb{C})$. We’re interested in certain finite dimensional simple $U$-modules $L(\lambda)$ indexed by partitions:

$$
\lambda = \begin{array}{c}
\text{(drawn as a collection of boxes piled up and to the left)}
\end{array}
$$

In particular, rectangular partitions:

$$
(a^c) = c
$$
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$$\lambda = \begin{array}{cccc}
 & & & \\
 & & & \\
 & & & \\
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(drawn as a collection of boxes piled up and to the left)

In particular, rectangular partitions:

$$(a^c) = c$$

$H_k$ has a commuting action with $U$ on the space

$$L((a^c)) \otimes L((b^d)) \otimes (L(\Box)) \otimes_k.$$
Centralizer properties

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(drawn as a collection of boxes piled up and to the left)

In particular, rectangular partitions:

The content of a box is its diagonal number.

$H_k$ has a commuting action with $U$ on the space

$L((a^c)) \otimes L((b^d)) \otimes (L(\square))^\otimes k$.

The content of a box is its diagonal number.
Centralizer properties

Let $U = U_q \mathfrak{gl}_n$ be the quantum group for $\mathfrak{gl}_n(\mathbb{C})$. We’re interested in certain finite dimensional simple $U$-modules $L(\lambda)$ indexed by partitions:

$\lambda = \begin{array}{cccc}
0 & 1 & 2 & 3 \\
-1 & & & \\
-2 & & &
\end{array}$

(drawn as a collection of boxes piled up and to the left)

In particular, rectangular partitions:

$(a^c) = c$

$H_k$ has a commuting action with $U$ on the space

$L((a^c)) \otimes L((b^d)) \otimes (L(\square))^k$.

The content of a box is its diagonal number.

The eigenvalues of $T_0$ and $T_k$ are controlled by the contents of addable boxes to $(a^c)$ and $(b^d)$.
Exploring \( L((a^c)) \otimes L((b^d)) \otimes (L(\square))^k \)

Products of rectangles:

\[
L((a^c)) \otimes L((b^d)) = \bigoplus_{\lambda \in \Lambda} L(\lambda) \tag{multiplicity one!}
\]

where \( \Lambda \) is the following set of partitions:
(Littlewood-Richardson, Okada)
Exploring $L((a^c)) \otimes L((b^d)) \otimes (L(\Box))^k$

Products of rectangles:

$$L((a^c)) \otimes L((b^d)) = \bigoplus_{\lambda \in \Lambda} L(\lambda)$$

(multiplicity one!)

where $\Lambda$ is the following set of partitions:

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Exploring $L((a^c)) \otimes L((b^d)) \otimes (L(\square))^{\otimes k}$

Products of rectangles:

$L((a^c)) \otimes L((b^d)) = \bigoplus_{\lambda \in \Lambda} L(\lambda)$ (multiplicity one!)

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Exploring \( L((a^c)) \otimes L((b^d)) \otimes (L(\square))^{\otimes k} \)

Products of rectangles:

\[
L((a^c)) \otimes L((b^d)) = \bigoplus_{\lambda \in \Lambda} L(\lambda) \quad \text{(multiplicity one!)}
\]

where \( \Lambda \) is the following set of partitions:
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$$L((a^c)) \otimes L((b^d)) = \bigoplus_{\lambda \in \Lambda} L(\lambda)$$

(multiplicity one!)

where $\Lambda$ is the following set of partitions...

(Littlewood-Richardson, Okada)
\( L \left( \begin{array}{c} \text{grid} \end{array} \right) \otimes L \left( \begin{array}{c} \text{grid} \end{array} \right) \)
$L \left( \begin{array}{ccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
8 & 9 & 10 & 11 & 12 & 13 & 14 \\
15 & 16 & 17 & 18 & 19 & 20 & 21 \\
22 & 23 & 24 & 25 & 26 & 27 & 28 \\
29 & 30 & 31 & 32 & 33 & 34 & 35 \\
36 & 37 & 38 & 39 & 40 & 41 & 42 \\
43 & 44 & 45 & 46 & 47 & 48 & 49 \\
50 & 51 & 52 & 53 & 54 & 55 & 56 \\
\end{array} \right) \otimes L \left( \begin{array}{ccc}
1 & 2 \\
\end{array} \right) \otimes L \left( \begin{array}{c}
\end{array} \right)$
$L \left( \begin{array}{ccc|ccc} 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 2 & 3 & 4 & 5 & 6 \\ \end{array} \right) \otimes L \left( \begin{array}{c|c} 1 & 2 \\ \hline 1 & 2 \\ \end{array} \right) \otimes L (\square) \otimes L (\square)$
$L \left( \begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & 2 \\ 1 & 3 & 5 \\ 2 & 4 & 1 \\ 3 & 2 & 4 \end{array} \right) \otimes L \left( \begin{array}{ccc} 1 & 2 \\ 3 & 4 \end{array} \right) \otimes L \left( \begin{array} { c } \square \end{array} \right) \otimes L \left( \begin{array} { c } \square \end{array} \right) \otimes L \left( \begin{array} { c } \square \end{array} \right)$

Shift by $1$ to the right.

Representations in tensor space are labeled by certain partitions $\lambda$. Basis labeled by tableaux from some partition $\mu$ in $(a-c) \otimes (b-d)$ to $\lambda$.

Calibrated: $Y_i$ acts by $t$ to the shifted content of box $i$. 

$\ast \ H_k$
$L \left( \begin{array}{|c|c|c|c|c|c|} \hline & & & & & \\
& & & & & \\
& & & & & \\
\hline \end{array} \right) \otimes L \left( \begin{array}{|c|c|} \hline & \\
& \hline \end{array} \right) \otimes L \left( \begin{array}{|c|} \hline \\
\hline \end{array} \right) \otimes L \left( \begin{array}{|c|} \hline \\
\hline \end{array} \right) \otimes L \left( \begin{array}{|c|} \hline \\
\hline \end{array} \right) \otimes L \left( \begin{array}{|c|} \hline \\
\hline \end{array} \right)$
\[ L \left( \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 \\ 16 & 17 & 18 & 19 & 20 \\ 21 & 22 & 23 & 24 & 25 \end{array} \right) \otimes L \left( \begin{array}{ccc} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \right) \otimes L (\square) \otimes L (\square) \otimes L (\square) \otimes L (\square) \otimes L (\square) \otimes L (\square) \otimes L (\square) \]
\[ L \left( \begin{array}{c|c|c|c|c|c|c} 
& & & & & & \\
\end{array} \right) \otimes \left( \begin{array}{c|c|c|c|c|c|c} 
& & & & & & \\
\end{array} \right) \otimes L(\square) \otimes L(\square) \otimes L(\square) \otimes L(\square) \otimes L(\square) \otimes L(\square) \otimes L(\square) \otimes L(\square) \]

\((\ast)\) \(H_k\) representations in tensor space are labeled by certain partitions \(\lambda\).
\[ L \left( \begin{array}{cc} 1 & 2 \\ 3 & 4 \\ \\ 5 \\ \\ 1 & 2 \\ 3 & 4 \\ \\ 5 \\ \\ \end{array} \right) \otimes L \left( \begin{array}{cc} 1 & 2 \\ 3 & 4 \\ \\ 5 \\ \\ 1 & 2 \\ 3 & 4 \\ \\ 5 \\ \\ \end{array} \right) \otimes L \left( \begin{array}{cc} 1 & 2 \\ 3 & 4 \\ \\ 5 \\ \\ 1 & 2 \\ 3 & 4 \\ \\ 5 \\ \\ \end{array} \right) \otimes L \left( \begin{array}{cc} 1 & 2 \\ 3 & 4 \\ \\ 5 \\ \\ 1 & 2 \\ 3 & 4 \\ \\ 5 \\ \\ \end{array} \right) \]

\((*)\) \(H_k\) representations in tensor space are labeled by certain partitions \(\lambda\).
$L \left( \begin{array}{cccc} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{array} \right) \otimes L \left( \begin{array}{cc} q & r \\ s & t \end{array} \right) \otimes L \left( \begin{array}{c} u \\ v \\ w \\ x \end{array} \right)$

(\ast) $H_k$ representations in tensor space are labeled by certain partitions $\lambda$.

(\ast) Basis labeled by tableaux from some partition $\mu$ in $(a^c) \otimes (b^d)$ to $\lambda$. 
\[ L \left( \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{array} \right) \otimes L \left( \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{array} \right) \otimes L \left( \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{array} \right) \otimes L \left( \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{array} \right) \otimes L \left( \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{array} \right) \otimes L \left( \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{array} \right) \otimes L \left( \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{array} \right) \]

(* \ H_k \) representations in tensor space are labeled by certain partitions \( \lambda \).

(* \ Basis labeled by tableaux from some partition \( \mu \) in \( (a^c) \otimes (b^d) \) to \( \lambda \).)

(* \ Calibrated \)
$L \left( \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & -1 & -2 & -3 & -4 & -5 \\ \end{array} \right) \otimes L \left( \begin{array}{cccccc} a & b \\ c & d \\ \end{array} \right) \otimes L (\square) \otimes L (\square) \otimes L (\square) \otimes L (\square) \otimes L (\square) \otimes L (\square)$

(∗) $H_k$ representations in tensor space are labeled by certain partitions $\lambda$.

(∗) Basis labeled by tableaux from some partition $\mu$ in $(a^c) \otimes (b^d)$ to $\lambda$.

(∗) Calibrated
\[
L \left( \begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
\end{array} \right) \otimes L \left( \begin{array}{cccc}
\end{array} \right) \otimes L \left( \begin{array}{c}
\end{array} \right) \otimes L \left( \begin{array}{c}
\end{array} \right) \otimes L \left( \begin{array}{c}
\end{array} \right) \otimes L \left( \begin{array}{c}
\end{array} \right) \otimes L \left( \begin{array}{c}
\end{array} \right)
\]

Shift by \( \frac{1}{2}(a - c + b - d) \)

\[\begin{array}{cccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 \\
\end{array} \]

\[\begin{array}{cccc}
-1 & -2 & -3 & -4 & -5 & -6 \\
\end{array} \]

\( (*) \) \( H_k \) representations in tensor space are labeled by certain partitions \( \lambda \).

\( (*) \) Basis labeled by tableaux from some partition \( \mu \) in \((a^c) \otimes (b^d)\) to \( \lambda \).

\( (*) \) Calibrated: \( Y_i \) acts by \( t \) to the shifted content of box\( i \).
From \{partitions in tensor space\} to \{box arrangements\}

\begin{array}{cccc}
1 & 3 \\
2 \\
4 \\
5 \\
\end{array}
From \{partitions in tensor space\} to \{box arrangements\}
From \{partitions in tensor space\} to \{box arrangements\}

\[\begin{array}{cccccc}
\hline
& & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
\hline
\end{array}\]

- \(\gamma(Y_1) = \tau_{45}\), \(\gamma(Y_2) = \tau_{35}\), \(\gamma(Y_3) = \tau_{r2}\), \(\gamma(Y_4) = \tau_{-25}\), \(\gamma(Y_5) = \tau_{-r2}\).

\(\square\) = boxes that must appear in the partition at level 0.
From \{partitions in tensor space\} to \{box arrangements\}

\[ \gamma(Y_1) = t^{4.5}, \quad \gamma(Y_2) = t^{3.5}, \quad \gamma(Y_3) = t^{r_2}, \quad \gamma(Y_4) = t^{-2.5}, \quad \gamma(Y_5) = t^{-r_2}. \]
From \{partitions in tensor space\} to \{box arrangements\}

\[
\gamma(Y_1) = t^{4.5}, \quad \gamma(Y_2) = t^{3.5}, \quad \gamma(Y_3) = t^{r_2}, \quad \gamma(Y_4) = t^{-2.5}, \quad \gamma(Y_5) = t^{-r_2}.
\]

\(\square\) = boxes that must appear in the partition at level 0.
From \{partitions in tensor space\} to \{box arrangements\}

\[\gamma(Y_1) = t^{4.5}, \gamma(Y_2) = t^{3.5}, \gamma(Y_3) = t^{r_2}, \gamma(Y_4) = t^{-2.5}, \gamma(Y_5) = t^{-r_2}.\]
From \{partitions in tensor space\} to \{box arrangements\}

\[ \gamma(Y_1) = t^{4.5}, \quad \gamma(Y_2) = t^{3.5}, \quad \gamma(Y_3) = t^{r_2}, \quad \gamma(Y_4) = t^{-2.5}, \quad \gamma(Y_5) = t^{-r_2}. \]

\[\square = \text{boxes that must appear in the partition at level 0.}\]
From \{partitions in tensor space\} to \{box arrangements\}

\[
\begin{array}{ccccc}
0 & 1 & 2 & 3 & \ldots \\
-1 & & & & \\
-2 & & & & \\
-3 & & & & \\
\ldots & -2 & 4 & & \\
-3 & -1 & & & \\
-1 & & & & \\
-5 & 1 & 3 & -4 & 2 \\
-5 & & & & \\
-2 & & & & \\
\end{array}
\]

\[\square = \text{boxes that must appear in the partition at level 0.}\]

\[\gamma(Y_1) = t^{4.5}, \gamma(Y_2) = t^{3.5}, \gamma(Y_3) = tr^2, \gamma(Y_4) = t^{-2.5}, \gamma(Y_5) = t^{-r^2}.\]
From \{partitions in tensor space\} to \{box arrangements\}

\[
\gamma(Y_1) = t^{4.5}, \quad \gamma(Y_2) = t^{3.5}, \quad \gamma(Y_3) = t^{r_2}, \quad \gamma(Y_4) = t^{-2.5}, \quad \gamma(Y_5) = t^{-r_2}.
\]

versus

\[
\gamma(Y_1) = t^{4.5}, \quad \gamma(Y_2) = t^{3.5}, \quad \gamma(Y_3) = t^{r_2}, \quad \gamma(Y_4^{-1}) = t^{2.5}, \quad \gamma(Y_5^{-1}) = t^{r_2}.
\]

\[\blacksquare = \text{boxes that must appear in the partition at level 0.}\]
Thanks!